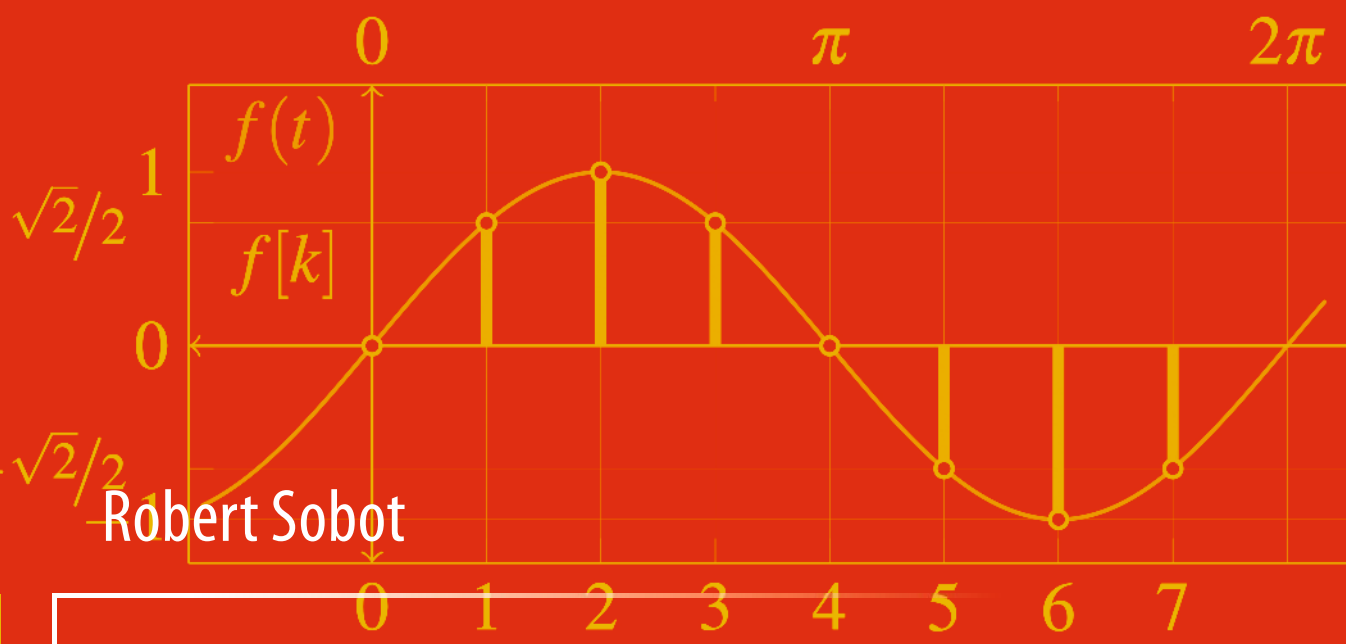

Robert Sobot

Engineering Mathematics by Example

Vol. III: Special Functions and Transformations

Second Edition

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ISBN 978-3-031-41202-8 ISBN 978-3-031-41203-5 (eBook)
https://doi.org/10.1007/978-3-031-41203-5

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To my math teacher Mr. Miloš Belušević

Preface

Preface to the Second Edition

It is inevitable that first edition of any type of textbook, and especially textbooks for mathematics, includes quite a few errors that slipped by all preproduction reviews. In this second edition, errors are thoroughly reviewed and corrected, with hopes that not many new ones are created.

As the original volume doubled, this second edition is split into three separate books, Vol. I, II, III. In order to reinforce natural development in the study, best effort is put to logically organize the presented examples and techniques so that subsequent problems reference the once already solved.

Île-de France, France — Robert Sobot
June 27, 2023

Preface to the First Edition

This tutorial book resulted from my lecture notes developed for undergraduate engineering courses in mathematics that I teach over the last several years at l'École Nationale Supérieure de l'Électronique et de ses Applications (ENSEA), Cergy in Val d'Oise department, France.

My main inspiration to write this tutorial type collection of solved problems came from my students who would often ask "How do I solve this? It is impossible to find the solution," while struggling to logically connect all the little steps and techniques that are required to combine together before reaching solution. In the traditional classical school systems mathematics used to be thought with the help of systematically organized volumes of problems that help us develop "the way of thinking." In other words, to learn how to apply the abstract mathematical concepts to everyday engineering problems. Same as for music, it is also true for mathematics that in order to reach high level of competence one must put daily effort into studying of typical forms over long period of time.

In this tutorial book I choose to give not only the complete solutions to the given problems, but also guided hints to techniques being used at the given moment. Therefore, problems presented in this book do not provide review of the rigorous mathematical theory, instead the theoretical background is assumed, while this set of classic problems provides a playground to play and to adopt some of the main problem-solving techniques.

The intended audience of this book are primarily undergraduate students in science and engineering. At the same time, my hope is that students of mathematics at any level will find this book to be useful source of practical problems to practice.

Île-de France, France Robert Sobot
November 30, 2020

Acknowledgments

I would like to acknowledge all those classic wonderful collections of mathematical problems that I grew up with and used as the source of my knowledge, and I would like to say thank you to their authors for providing me with thousands of problems to work on. Specifically, I would like to acknowledge classic collections written in ex–Yugoslavian, Russian, English and French languages, some are listed in the bibliography. Hence, I do want to acknowledge their contributions that are clearly visible throughout this book, which are now being passed on to my readers.

I would like to thank all of my former and current students who I had opportunity to tutor in mathematics since my student years. Their relentless stream of questions posed with unconstrained curiosity forced me to open my mind, to broaden my horizons, and to improve my own understandings.

I want to acknowledge Mr. Allen Sobot for reviewing and verifying some of the problems, for very useful discussions and suggestions, as well as his work on technical redaction in first edition of this book.

Sincere gratitude goes to my publisher and editors for their support and making this book possible.

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Acronyms

$f(x)$, $g(x)$	Functions of x
$f'(x)$, $g'(x)$	First derivatives of functions of $f(x)$, $g(x)$
$f(g(x))$	Composite function
$\sin(x)$	Sine of x
$\cos(x)$	Cosine of x
$\tan(x)$	Tangent of x
$\arctan(x)$	Arctangent of x
$\text{sinc}\,(t)$	Sine cardinal of t
$\log(x)$	Logarithm of x, base 10
$\ln(x)$	Natural logarithm of x, base e
$\log_n(x)$	Logarithm of x, base n
$x * y$	Convolution product
i	Imaginary unit, $i^2 = -1$
j	Imaginary unit, $j^2 = -1$
$\Im(z)$	Imaginary part of a complex number z
$\Re(z)$	Real part of a complex number z
$\mathscr{F}(x(t))$	Fourier transform of $x(t)$
$X(\omega)$	Fourier transform of $x(t)$
$\mathscr{L}^{-1}\{F(s)\}$	Inverse Laplace transform of $F(s)$
$\mathscr{F}^{-1}(X(\omega))$	Inverse Fourier transform of $X(\omega)$
$\overset{F}{\rightarrow}$	Apply Fourier transform
$\overset{LT}{\longrightarrow}$	Apply Laplace transform
$\overset{\text{l'H}}{=}$	Apply l'Hôpital rule
DFT	Discrete Fourier transform
FFT	Fast Fourier transform
$\delta(x)$	Dirac distribution
$\Lambda(x)$	Triangular distribution
$\Pi(x)$	Square (rectangular) distribution
Ш_T	Dirac comb whose period is T
$\Gamma(t)$	Heaviside step function of t
$u(t)$	Heaviside step function of t
$r(t)$	Ramp function of t
$\text{sign}\,(t)$	Sign function of t
(a, b)	Pair of numbers
(a, b)	Open interval
$\equiv$	Equivalence

$\{a_1, a_2, a_3, \dots\}$	List of elements, vector, series
$\{a_n\}$	List of elements, vector, series
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$a \in \mathbb{C}$	Number a is included in the set of complex numbers
$a \in \mathbb{R}$	Number a is included in the set of real numbers
$\forall x$	For all values of x
$\angle$	Angle, argument
$\Rightarrow$	It follows that
$\frac{dy}{dx}$	Derivative of y relative to x
u'_x	Partial derivative of $u(x, y, z, \dots)$ relative to x
$u''_{x,y}$	Second partial derivative of $u(x, y, z, \dots)$, relative to x then to y
z^*	Complex conjugate of number z
$\|x\|$	Absolute value of x
$A_{m,n}$	Matrix A whose size is $m \times n$
I_n	The identity square matrix whose size is $n \times n$
$\|A\|$	Determinant of matrix A
A^T	Transpose of matrix A
Δ	Difference between two variables
Δ	Main determinant of a matrix
Δ_x	Cramer's sub-determinant relative to variable x
$\vec{a}$	Vector a
$\sum_{i=0}^{\infty} a_i$	Sum of elements $\{a_0, a_1, \dots, a_\infty\}$
dB	Decibel
dBm	Decibel normalized to "10^{-3}", i.e. "milli"
$H(jx)$	Transfer function
$x \to \underset{\twoheadleftarrow}{a}$	Limiting to a from its right side (x-axis)
$x \to \underset{\twoheadrightarrow}{a}$	Limiting to a from its left side (x-axis)
$x \to a\downarrow$	Limiting to a from above (y-axis)
$x \to a\uparrow$	Limiting to a from below (y-axis)
$\{\cdot\}$	in-line hints

Series

1

Infinite series: an infinite ordered list of elements (terms) to be added

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

Limit of infinite series: first, the sum of finite number n of elements a_i, and then limit n to infinity

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Arithmetic progression: a sequence of numbers where the difference d between any two subsequent terms is constant. That being case, given the first term a_1, a general term a_n is found as

$$a_n = a_1 + d\,(n-1)$$

The sum of the first n terms in arithmetic progression

$$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$$

Geometric progression: a sequence of numbers where the *ratio* r between any two subsequent terms is constant. Given the first term a_1, a general term a_n is found as

$$a_1,\ \ a_2 = a_1\, r,\ \ \ a_3 = a_2\, r, \ldots$$
$$a_n = a_1\, r^{n-1}$$

The sum of geometric progression: if $a_1 = a,\ (a \neq 0)$ then the sum of first n terms is found as

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_1

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a\, r^{k-1} = \underbrace{ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1}}_{\text{n terms}}$$

Assuming $a \neq 0$, there are two main cases:

1. $r = 1$: the sum of first n elements is

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a\, \cancel{r^{k-1}}^{1} = n\, a$$

thus, the sum of infinite series is

$$\sum_{n=1}^{n\to\infty} a\, r^{n-1} = \sum_{n=1}^{n\to\infty} a\, \cancel{1^{n-1}}^{1} = \sum_{n=1}^{n\to\infty} a = \underbrace{a + a + a + \cdots}_{\text{infinitely terms}} \to \infty$$

In conclusion, if $r = 1$, infinite geometric series is divergent.

2. $r \neq 1$: The sum of first n elements is derived as follows:

$$\begin{aligned} S_n &= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} \quad \therefore \\ r\, S_n &= \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} + ar^n \\ (S_n - r\, S_n) &= a - ar^n \quad \Rightarrow \quad S_n = \frac{a\,(1 - r^n)}{1 - r} \end{aligned}$$

Convergence of the geometric sum: depending upon the value of $|r| \neq 0$, where a is the first term in the series, then the infinite sum may be calculated as

(a) $|r| > 1$: in this case, each power term r^n of the geometric series is greater and greater; therefore

$$\sum_{n=1}^{\infty} a\, r^{n-1} = \lim_{n\to\infty} \frac{a\,(1 - r^n)}{1 - r} (=) \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{-anr^{n-1}}{-1} \to \infty$$

In conclusion, for $|r| >, 1$ infinite geometric series is divergent.

(b) $|r| < 1$: in this case, recall that powers r^n of number r inferior to one are smaller and smaller tending to zero; thus

$$\sum_{n=1}^{\infty} a\, r^{n-1} = \lim_{n\to\infty} \frac{a\left(1 - \cancel{r^n}^{0}\right)}{1 - r} = \frac{a}{1 - r}$$

In conclusion, for $|r| < 1$, infinite geometric series is convergent.

Mathematical induction: given identity $F(n)$ in function of n can be proven by the mathematical induction technique that consists of three steps:

1. $F(n)$ must be proven to be true for $n = 1$.
2. Assumption is that $F(k)$ is also true, where k is an arbitrary value of $n = k$.
3. $F(n)$ must be proven to be true for the *next* value of $n = k + 1$

and then, the conclusion is that given identity $F(n)$ is true for all n.

Problems

Finite or infinite series are summed in accordance with their respective forms. In general, it is important to first determine if the series is in the arithmetic or geometric form and then to apply various algebraic methods.

1.1 Sequence Notation

Determine general term a_n of infinite series in P.1.7 to P.1.9.

1.1. $1, 3, 7, 15, 31, 63, 127 \cdots$ **1.2.** $1, -1, 1, -1, 1, \cdots$ **1.3.** $S = 2^3 + 3^3 + \cdots + n^3$

1.4. $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \cdots$ **1.5.** $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \cdots$ **1.6.** $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$

1.7. $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \cdots$ **1.8.** $S = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$ **1.9.** $1, \frac{2}{1\cdot 2}, \frac{3}{1\cdot 2\cdot 3}, \frac{4}{1\cdot 2\cdot 3\cdot 4}, \cdots$

Calculate sums of finite series P.1.10 to P.1.15.

1.10. $\sum_{i=1}^{4} i$ **1.11.** $\sum_{i=1}^{4} 2$ **1.12.** $\sum_{i=1}^{3} i^2$

1.13. $\sum_{j=0}^{5} 2^j$ **1.14.** $\sum_{k=1}^{4} \frac{1}{k}$ **1.15.** $\sum_{n=1}^{3} \frac{n-1}{n^2+3}$

1.2 Finite Series

Derive sums in P.1.16 to P.1.21.

1.16. $\sum_{i=1}^{n} 1$ **1.17.** $\sum_{i=1}^{n} i$ **1.18.** $\sum_{n=1}^{5} \frac{2^{n+2}}{3^n}$

1.19. $\sum_{i=1}^{n} i(4i-3)$

1.20. $\sum_{i=1}^{n} \frac{1}{2^k}$

1.21. $\frac{5}{6}, \frac{11}{12}, \frac{23}{24}, \frac{47}{48}, \cdots$

1.22. $\sum_{i=1}^{25} (5-2i)$

1.23. $\sum_{n=1}^{10} (-2)^{n-1}$

1.24. $\sum_{k=1}^{n} (-1)^{k-1} x^{2k}$

1.3 Infinite Series

Derive infinite sums in P.1.25 to P.1.30.

1.25. $\sum_{n=3}^{\infty} \sqrt{n-3}$

1.26. $\lim_{k\to\infty} \sum_{n=0}^{11} \cos \frac{k n\pi}{6}$

1.27. $\lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{3}{n} \left[\left(\frac{i}{n} \right)^2 + 1 \right]$

1.28. $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \right)$

1.29. $\lim_{n\to\infty} \left[\left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) \right]$

1.30. $\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

1.4 Geometric Series

Deduce general terms of series in P.1.31 to P.1.38 and then calculate the sums.

1.31. $\sum_{n=1}^{\infty} -2 \left(\frac{3}{4} \right)^{n-1}$

1.32. $\sum_{1}^{5} 5(-5)^n$

1.33. $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots$

1.34. $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

1.35. $S = \sum_{n=1}^{\infty} 2^{2n}\, 3^{1-n}$

1.36. $\sum_{k=0}^{10} (4, -8, 16, -32, 64, \ldots)$

1.37. $S = \sum_{n=1}^{\infty} x^{n-1}; \quad |x| < 1$

1.38. $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{3^{n-1}}$

1.5 Mathematical Induction

Prove identities in P.1.39 to P.1.45.

1.39. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

1.40. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

1.41. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

1.42. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

1.43. $1 + 3 + 5 + \cdots + (2n-1) = n^2$

1.44. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

1.45. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

Answers

1.1 Sequence Notation

1.1. One possible set of solutions may be derived by noticing that all terms of given series are close to powers of two series, i.e. $2^n = 2, 4, 8, 16, 32, \ldots$, thus

$$1, 3, 7, 15, 31, 63, 127 \cdots \;\Rightarrow\; a_k = 2^k - 1, \quad k \in (1, \infty)$$

or,

$$a_k = 2^{k+1} - 1, \quad k \in (0, \infty), \quad \text{etc.}$$

1.2. Series whose elements are alternating ± 1 may be written as

$$1, -1, 1, -1, 1, \cdots \;\Rightarrow\; a_m = (-1)^m, \quad m \in (2, \infty),$$

or,

$$a_m = (-1)^{m-1}, \quad m \in (1, \infty), \quad \text{or,}$$

$$a_m = \cos(m\pi), \quad m \in (1, \infty), \quad \text{or,}$$

$$a_m = e^{j\,m\pi}, \quad (j^2 = -1) \quad \text{and} \quad m \in (1, \infty), \quad \text{etc.}$$

1.3. Probably most obvious response is to write,

$$S = 2^3 + 3^3 + \cdots + n^3 = \sum_{k=2}^{n} k^3$$

However, all *indexes* may be reduced, for example, by one without changing the total number of terms nor the total sum, as

$$S = \sum_{j=1}^{n-1} (j + 1)^3 = (1 + 1)^3 + (2 + 1)^3 + (3 + 1)^3 + \cdots + ((n - 1) + 1)^3$$
$$= 2^3 + 3^3 + \cdots + n^3$$

Or, if for any reason it is preferred that the index counter starts with zero, then

$$S = \sum_{m=0}^{n-2} (m + 2)^3 = (0 + 2)^3 + (1 + 2)^3 + (2 + 2)^3 + \cdots + ((n - 2) + 2)^3$$
$$= 2^3 + 3^3 + \cdots + n^3$$

1.4. One possible set of relations between numerator and denominator may be $2^n - 1$, that is

$$1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \cdots \;\Rightarrow\; a_n = \frac{n}{2n - 1}, \quad n \in (1, \infty), \quad \text{or,}$$

$$a_n = \frac{n+1}{2n+1}, \quad n \in (0, \infty), \quad \text{etc.}$$

1.5. Denominator is product of two subsequent numbers, thus

$$\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \cdots \Rightarrow a_k = \frac{1}{k(k+1)}, \quad k \in (1, \infty) \quad \text{or,}$$

$$a_k = \frac{1}{(k+1)(k+2)}, \quad k \in (0, \infty) \quad \text{or,}$$

$$a_k = \frac{1}{(k-1)\,k}, \quad k \in (2, \infty) \quad \text{etc.}$$

1.6. Given series may be written as

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots = \frac{+1}{1}, \frac{-1}{2}, \frac{+1}{3}, \frac{-1}{4}, \cdots = \{\text{see A.1.2}\}$$

$$\Rightarrow a_n = \frac{(-1)^{n+1}}{n}, \quad n \in (1, \infty)$$

It is worth mentioning that this infinite sum converges to $\ln 2$.

1.7. Note that general term of any given series may be written in infinite number of ways, there is no unique solution. Some forms seem more evident than the others, nevertheless there are many possibilities. For example, it may seem natural to say that numerator of any term equal denominator plus one, i.e.

$$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \cdots \Rightarrow a_i = \frac{i+1}{i}, \quad i \in (2, \infty)$$

or,

$$a_i = \frac{i+2}{i+1}, \quad i \in (1, \infty) \quad \text{or,} \quad a_i = \frac{i}{i-1}, \quad i \in (3, \infty), \cdots$$

1.8. By inspection, denominator equal numerator plus one, thus general form of this sum may be written in any of the following forms

$$S = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+1} = \sum_{k=0}^{\infty} \frac{k+1}{k+2} = \sum_{m=-1}^{\infty} \frac{m+2}{m+3}$$

or relative to any other index starting at arbitrary value.

1.9. Denominator is expanding in the form of factorial, thus

$$1, \frac{2}{1\cdot 2}, \frac{3}{1\cdot 2\cdot 3}, \frac{4}{1\cdot 2\cdot 3\cdot 4}, \cdots \Rightarrow a_k = \frac{k}{k!}, \quad k \in (1, \infty)$$

1.10. Finite number of terms $k = 4$ in series can be added in a finite amount of time. Thus, a four element sequence where each term equal to its position in the series, here a general term $n_i = i$, i.e.

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$, adds up as

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

This general form of a sequence is better known as "arithmetic progression" or, in this case, the sum of first m integers, where the number of terms to be added $m = 4$. Note that indexing of terms starts with "1" and finishes with $m = 4$, by consequence there the total of m terms to be added.

1.11. A finite sequence of k identical terms, here a general term $n_i = 2$ (that is to say, not a function of i), i.e. $n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2$, is added as

$$\sum_{i=1}^{4} 2 = \underbrace{2 + 2 + 2 + 2}_{4 \text{ times}} = 4 \cdot 2 = 8$$

where, the indexing range is $i \in (1, 4)$. As index of the last element in the series is '4' by consequence there is the total of $k = 4$ terms to be added.

1.12. Sum of the first $k = 3$ integer squares $n_i = i^2$, i.e. $n_1 = 1^2, n_2 = 2^2, n_3 = 3^2$, is

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

where, the indexing range is $i \in (1, 3)$. As index of the last element in the series is '3' by consequence there is the total of $k = 3$ terms (i.e. 1, 4, and 9) to be added.

1.13. Finite series of "powers of two" $a_j = 2^j$, i.e. $a_0 = 2^0, a_1 = 2^1, a_2 = 2^2, a_3 = 2^3, a_4 = 2^4, a_5 = 2^5$, is added as

$$\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63$$

where, the indexing range is $j \in (0, 5)$. As index of the last element in the series is '$n = 5$' and index of the first element is '0', by consequence there is the total of $k = n + 1 = 6$ terms (i.e. 1, 2, 4, 8, 16 and 32) to be added. That is to say, elements indexed one to five, plus the one with the index zero.

1.14. Finite series of the first $n = 4$ integer fractions, $a_1 = 1/1, a_2 = 1/2, a_3 = 1/3, a_4 = 1/4$, sums as

$$\sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

where, the indexing range is $k \in (1, 4)$. As index of the last element in the series is '$n = 4$' and index of the first element is '1', by consequence there is the total of $n = 4$ terms to be added.

1.15. Given finite series of three elements in the form

$$a_i = \frac{n-1}{n^2+3}$$

$\therefore$ its sum is

$$\sum_{n=1}^{3} a_i = \sum_{n=1}^{3} \frac{n-1}{n^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = \frac{0}{3} + \frac{1}{7} + \frac{2}{12} \frac{1}{6} = \frac{13}{42}$$

1.2 Finite Series

1.16. In the general case of the last index value n, the sum S is derived in function of n as

$$S = \sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n\times 1} = n$$

because, a general term of this series is $a_i = 1$, that is to say, $a_1 = 1, a_2 = 1, a_3 = 1, \ldots$ repeated n times. Note that this solution is general because n may be finite or infinite. In the case that this series is infinite, then the sum is found as limit

$$\lim_{n\to\infty} S = \lim_{n\to\infty} n = \infty$$

1.17. In comparison with P.1.10, general solution (a.k.a. Gauss' formula) of this sum is derived in function of n. The legend has it that this classic problem of the sum of an arithmetic sequence was solved by Gauss when he was still in the elementary school. His proof takes advantage of the commutative property of addition by writing the same sequence in both ascending and descending order and creating n identical summing terms as

$$\begin{array}{llllllll} S = & 1 & +2 & +3 & +\cdots & +(n-1) & +n \\ S = & n & +(n-1) & +\cdots & +3 & +2 & +1 \\ \hline \therefore & & & & & & \end{array}$$

$$2S = \underbrace{(n+1) \quad +(n+1) \quad +(n+1) \quad +\cdots \quad +(n+1) \quad +(n+1)}_{n\times(n+1)}$$

because vertical sum of each pair results in $(n+1)$ repeated n times: $(n+1)$, $2+(n-1) = (n+1)$, etc. Thus, double sum is

$$2S = n(n+1) \quad \Rightarrow \quad S = \frac{n(n+1)}{2}$$

In conclusion, sum of the first n integers is

$$S = \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

1.18. $$\sum_{n=1}^{5}\frac{2^{n+2}}{3^n}=\sum_{n=1}^{5}\frac{2^n2^2}{3^n}=4\sum_{n=1}^{5}\left(\frac{2}{3}\right)^n=4\left(\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^5\right)$$
$$=4\left(\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}\right)=4\,\frac{422}{243}$$

1.19. This sum may be split into two already known sums as

$$\sum_{i=1}^{n}i(4i-3)=\sum_{i=1}^{n}4i^2-\sum_{i=1}^{n}3i=4\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i=\{\text{see P.1.40 and P.1.17}\}$$
$$=4\frac{n(n+1)(2n+1)}{6}-3\frac{n(n+1)}{2}=\frac{4n(n+1)(2n+1)-9n(n+1)}{6}$$
$$=(n+1)\,\frac{8n^2-5n}{6}=\frac{n(n+1)(8n-5)}{6}$$

1.20. When $n\to\infty$ this series is known as description of Zeno's paradox. Noninfinite sum may be solved as

$$S_n=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}$$
$$\therefore$$
$$2S_n=\frac{2}{2}+\frac{2}{2^2}+\frac{2}{2^3}+\cdots+\frac{2}{2^{n-1}}+\frac{2}{2^n}=1+\underbrace{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}}_{S_n}-\frac{1}{2^n}$$
$$=1+S_n-\frac{1}{2^n}\quad\Rightarrow\quad \underline{S_n=1-\frac{1}{2^n}}$$

Note that for $n\to\infty$ this sum converges to one.

1.21. General term of the given series may be deduced as

$$\frac{5}{6},\frac{11}{12},\frac{23}{24},\frac{47}{48},\cdots\quad\therefore\quad \frac{5}{6}=\frac{6-1}{6}=1-\frac{1}{6}=1-\frac{1}{3\cdot 2^1}$$
$$\frac{11}{12}=\frac{12-1}{12}=1-\frac{1}{12}=1-\frac{1}{3\cdot 2^2}$$
$$\frac{23}{24}=\frac{24-1}{24}=1-\frac{1}{24}=1-\frac{1}{3\cdot 2^3}$$
$$\frac{27}{48}=\frac{48-1}{48}=1-\frac{1}{48}=1-\frac{1}{3\cdot 2^4}$$
$$\cdots$$
$$a_k=1-\frac{1}{3}\,\frac{1}{2^k}$$

so that,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(1-\frac{1}{3}\frac{1}{2^k}\right) = \sum_{k=1}^{n} 1 - \frac{1}{3}\sum_{k=1}^{n}\frac{1}{2^k} = \{\text{see A.1.20}\} = n - \frac{1}{3}\left(1-\frac{1}{2^n}\right)$$

1.22. $$\sum_{i=1}^{25}(5-2i) = \sum_{i=1}^{25} 5 - 2\sum_{i=1}^{25} i = \left\{\sum_{k=1}^{n} a_k = \frac{n}{2}(a_1+a_n)\right\}$$
$$= 25\cdot 5 - 2\frac{25}{2}(1+25) = 25\cdot 5 - 25 - 25\cdot 25 = 25(5-1-25) = -525$$

1.23. $$\sum_{n=1}^{10}(-2)^{n-1} = \sum_{n=1}^{10}(-1)^{n-1}2^n\,2^{-1} = \frac{1}{2}\sum_{n=1}^{10}(-1)^{n-1}2^n$$
$$= \frac{1}{2}\left((-1)^{1-1}2^1 + (-1)^{2-1}2^2 + (-1)^{3-1}2^3\cdots + (-1)^{10-1}2^{10}\right)$$
$$= \frac{1}{2}\left(2-4+8-16+32-64+128-256+512-1024\right)$$
$$= 1-2+4-8+16-32+64-128+256-512 = -341$$

1.24. $$\sum_{k=1}^{n}(-1)^{k-1}x^{2k} = (-1)^{1-1}x^{2(1)} + (-1)^{2-1}x^{2(2)} + (-1)^{3-1}x^{2(3)} + \cdots + (-1)^{n-1}x^{2n}$$
$$= (-1)^0x^2 + (-1)^1x^4 + (-1)^2x^6 + \cdots + (-1)^{n-1}x^{2n}$$
$$= x^2 - x^4 + x^6 - \cdots + (-1)^{n-1}x^{2n}$$

1.3 Infinite Series

1.25. $S = \sum_{n=3}^{\infty}\sqrt{n-3} = 0+1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n-3}+\cdots$

Obviously, this infinite sum is *divergent* because the last term in the sum is infinite, $\sqrt{\infty-3}=\infty$

1.26. Terms of this series in the first period are

$$S = \sum_{n=0}^{11}\cos\frac{n\pi}{6} = \cos\frac{0\pi}{6} + \cos\frac{1\pi}{6} + \cos\frac{2\pi}{6} + \cos\frac{3\pi}{6} + \cos\frac{4\pi}{6} + \cos\frac{5\pi}{6}$$
$$+ \cos\frac{6\pi}{6} + \cos\frac{7\pi}{6} + \cos\frac{8\pi}{6} + \cos\frac{9\pi}{6} + \cos\frac{10\pi}{6} + \cos\frac{11\pi}{6}$$
$$= \underbrace{1+\frac{\sqrt{3}}{2}+\frac{1}{2}+0-\frac{1}{2}-\frac{\sqrt{3}}{2}-1-\frac{\sqrt{3}}{2}-\frac{1}{2}-0+\frac{1}{2}+\frac{\sqrt{3}}{2}}_{\sum=0}$$
$$= 0$$

Therefore, when $k \to \infty$ this sum is extended to *infinite number of full periods*, thus the total sum is zero (i.e. series is convergent). In any other case, the infinite sum of this series is divergent.

1.27. Note which variable is the sum counter, and which one is the limit counter,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{\infty} \left[\frac{3}{n}\frac{i^2}{n^2} + \frac{3}{n}\right] = \lim_{n\to\infty} \sum_{i=1}^{\infty} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3} \sum_{i=1}^{\infty} i^2 + \frac{3}{n} \sum_{i=1}^{\infty} 1\right] \\
&= \{\text{see P.1.40 and P.1.16}\} \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} n\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2} \frac{n+1}{n} \frac{2n+1}{n} + 3\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2} \frac{n\left(1+\frac{1}{n}\right)}{n} \frac{n\left(2+\frac{1}{n}\right)}{n} + 3\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\left(1+\overset{0}{\cancel{\frac{1}{n}}}\right)\left(2+\overset{0}{\cancel{\frac{1}{n}}}\right) + 3\right] \\
&= \frac{1}{2}(1+0)(2+0) + 3 \\
&= 4
\end{aligned}$$

1.28. Given series may be rewritten so that

$$\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) = \lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \{\text{see P.1.17}\} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{n\to\infty} \frac{n^{\cancel{2}}\left(1+\overset{0}{\cancel{\frac{1}{n}}}\right)}{2n^{\cancel{2}}} = \frac{1}{2}$$

1.29. Given series may be factorized so that

$$\begin{aligned}
\lim_{n\to\infty} \left[\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)\right] &= \lim_{n\to\infty} \left(\frac{4-1}{4}\,\frac{9-1}{9}\,\cdots\,\frac{n^2-1}{n^2}\right) \\
&= \{a^2 - b^2 = (a-b)(a+b)\} \\
&= \lim_{n\to\infty} \left(\frac{(2-1)(2+1)}{2^2}\,\frac{(3-1)(3+1)}{3^2}\,\cdots\,\frac{(n-1)(n+1)}{n^2}\right)
\end{aligned}$$

$$= \lim_{n\to\infty} \left(\frac{1\cdot 3}{2\cdot 2} \frac{2\cdot 4}{3\cdot 3} \frac{3\cdot 5}{4\cdot 4} \frac{4\cdot 6}{5\cdot 5} \cdots \frac{(n-2)\,n}{(n-1)\cdot(n-1)} \frac{(n-1)\,(n+1)}{n\cdot n} \right)$$

$$= \lim_{n\to\infty} \frac{n+1}{2n} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

1.30. One possible method to resolve limits of series similar to

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

is to use partial fractions decomposition method in such a way that sum of the resulting repetitive pattern cancels with possible exception of a constant, e.g. '1'. The repetitive pattern may be deduced by decomposing a few last terms in the series, which (if exists) may be extended back to the leading terms as

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2\,n} - \frac{1}{(n+1)} + \frac{1}{2\,(n+2)}$$

$$\frac{1}{(n-1)n(n+1)} = \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2\,(n+1)}$$

$$\frac{1}{(n-2)(n-1)n} = \frac{1}{2(n-2)} - \frac{1}{(n-1)} + \frac{1}{2\,n}$$

$$\cdots$$

The emerging pattern may be applied to the leading terms (and verified if still correct) as

$$\cdots$$

$$\frac{1}{6\cdot 7\cdot 8} = \frac{1}{2\cdot 6} - \frac{1}{7} + \frac{1}{2\cdot 8} \quad \checkmark$$

$$\frac{1}{5\cdot 6\cdot 7} = \frac{1}{2\cdot 5} - \frac{1}{6} + \frac{1}{2\cdot 7} \quad \checkmark$$

$$\frac{1}{4\cdot 5\cdot 6} = \frac{1}{2\cdot 4} - \frac{1}{5} + \frac{1}{2\cdot 6} \quad \checkmark$$

$$\frac{1}{3\cdot 4\cdot 5} = \frac{1}{2\cdot 3} - \frac{1}{4} + \frac{1}{2\cdot 5} \quad \checkmark$$

$$\frac{1}{2\cdot 3\cdot 4} = \frac{1}{2\cdot 2} - \frac{1}{3} + \frac{1}{2\cdot 4} \quad \checkmark$$

$$\frac{1}{1\cdot 2\cdot 3} = \frac{1}{2\cdot 1} - \frac{1}{2} + \frac{1}{2\cdot 3} \quad \checkmark$$

As a consequence, the sum simplifies as

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{(n+1)} + \frac{1}{2(n+2)}$$
$$+ \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$$
$$+ \frac{1}{2(n-2)} - \frac{1}{(n-1)} + \frac{1}{2n}$$
$$\cdots$$
$$+ \frac{1}{12} - \frac{1}{7} + \frac{1}{16}$$
$$+ \frac{1}{10} - \frac{1}{6} + \frac{1}{14}$$
$$+ = \frac{1}{8} - \frac{1}{5} + \frac{1}{12}$$
$$+ \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$$
$$+ \frac{1}{4} - \frac{1}{3} + \frac{1}{8}$$
$$+ \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$$

That is to say, the series sum consists of non–canceled terms and converges as

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2(n+1)}^{0} - \frac{1}{(n+1)}^{0} + \frac{1}{2(n+2)}^{0} \right) = \frac{1}{4}$$

In conclusion, this series is convergent, to 1/4.

1.4 Geometric Series

1.31. First few terms of this infinite series where $n \in (1, \infty)$ are

$$a_n : \quad -2, -\frac{3}{2}, -\frac{9}{8}, -\frac{27}{32}, \cdots -2\left(\frac{3}{4}\right)^{n-1} \cdots$$

Therefore, the sum of infinite series where first term in the series is $a_1 = -2$, $r = 3/4 < 1$ is

$$\sum_{n=1}^{\infty} -2\left(\frac{3}{4}\right)^{n-1} = \frac{-2}{1-\left(\frac{3}{4}\right)} = -8$$

Note that indexing starts with $n = 1$.

1.32. Property of geometric series is that ratio r of any two subsequent terms is constant, i.e.

$$r = \frac{a_{n+1}}{a_n} = \frac{5(-5)^{n+1}}{5(-5)^n} = -5^{n+1-n} = -5 \ \texttt{(const.)}$$

First few terms of this infinite series where $n \in (1, \infty)$ are

$$a_n : \quad -25, 125, \cdots 5(-5)^n \cdots$$

then, the first term in the series is $a_1 = 5(-5)^1 = -25$ and $|-5| = 5 > 1$. Sum of the first $n = 5$ terms of geometric progression is

$$S_5 = \sum_{n=1}^{5} 5(-5)^n = \left\{ \frac{a_1(1-r^n)}{1-r} \right\} = \frac{-25\left(1-(-5)^5\right)}{1-(-5)} = \frac{-78\,150}{6} = -13\,025$$

1.33. Given

$$\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots$$

Property of geometric series is that ratio r of any two subsequent terms is constant, i.e.

$$r = \frac{1/2}{3/2} = \frac{1/6}{1/2} = \frac{1/18}{1/6} = \cdots = \frac{1}{3} \ \texttt{(const.)}$$

then, the first term in the series is for $n = 0$, i.e. $a_0 = 3/2$. Therefore, the first few terms are

$$a_0 = \frac{3}{2}\left(\frac{1}{3}\right)^0 = \frac{3}{2}; \quad a_1 = \frac{3}{2}\left(\frac{1}{3}\right)^1 = \frac{1}{2}; \quad a_2 = \frac{3}{2}\left(\frac{1}{3}\right)^2 = \frac{1}{6}; \quad \cdots a_n = \frac{3}{2}\left(\frac{1}{3}\right)^n;$$

Infinite geometric progression is convergent under condition

$$|r| < 1 \quad \Rightarrow \left|\frac{1}{3}\right| < 1 \ \checkmark$$

therefore, sum of infinite geometric progression is

$$S_\infty = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{1}{3}\right)^n = \left\{ \frac{a_0}{1-r} \right\} = \frac{3/2}{1-1/3} = \frac{9}{4}$$

Note that indexing starts with $n = 0$.

1.34. Given infinite series,

$$S = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

it follows that

1. ratio of subsequent terms is

$$\frac{-\frac{\cancel{10}\,2}{3}}{\frac{\cancel{5}}{1}} = \frac{\frac{\cancel{20}\,2}{\cancel{9}\,3}}{-\frac{\cancel{10}}{\cancel{3}}} = \frac{-\frac{\cancel{40}\,2}{\cancel{27}\,3}}{\frac{\cancel{20}}{\cancel{9}}} = \cdots = -\frac{2}{3} = r \quad \Rightarrow \quad |r| = \frac{2}{3} < 1$$

2. then given series may be written as

$$S = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = 5\left(-\frac{2}{3}\right)^0 + 5\left(-\frac{2}{3}\right)^1 + 5\left(-\frac{2}{3}\right)^2 + \cdots + 5\left(-\frac{2}{3}\right)^n$$

3. therefore, sum of infinitely long series where $r = -2/3$ and the first term in the series is $a_0 = 5$ is calculated as

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n = \left\{\frac{a_0}{1-r}\right\} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{\cancel{5}}{\frac{\cancel{5}}{3}} = 3$$

Note that indexing starts at $n = 0$.

1.35. Given expression may be converted into the general form of geometric series as

$$S = \sum_{n=1}^{\infty} 2^{2n}\, 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n\, 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{4 \times 4^{n-1}}{3^{n-1}}$$
$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1} \quad \therefore \quad |r| = \frac{4}{3} > 1$$

Therefore, since $|r| > 1$ the conclusion is that this series is divergent, i.e. $S \to \infty$.

1.36. Given series $4, -8, 16, -32, 64, \ldots$ Property of geometric series is that ratio r of any two subsequent terms is constant, i.e.

$$r = \frac{-8}{4} = \frac{16}{-8} = \frac{-32}{16} = -2 \ \texttt{(const.)}$$

then, given series may be written as

$$4 - 8 + 16 - 32 + 64 + \cdots = 4(-2)^0 + 4(-2)^1 + 4(-2)^2 + 4(-2)^3 + 4(-2)^4 + \cdots + 4(-2)^n$$

therefore, $r = -2$ and the first term is for $n = 0$, i.e. $a_0 = 4$. Sum of the first $n = 1$ to $n = 10$ terms of geometric progression is

$$S_{10} = \sum_{n=0}^{10} 4(-2)^n = \frac{-2\left(1 - (-2)^{10}\right)}{1 - (-2)} = 2\,732$$

Note that indexing starts at $n = 0$.

1.37. This expression is in the general form of geometric series, then given condition that $|x| < 1$, it follows that

$$S = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=1}^{\infty} 1\, x^{n-1} = \{a_1 = 1,\ r = x\} = \frac{1}{1-x}$$

1.38. Given infinite geometric series may be written as

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}}{3^{n-1}} = \sum_{1}^{\infty} \left(\frac{-1}{3}\right)^{n-1}$$

first few terms of this infinite series where $n \in (1, \infty)$ are

$$a_n : \quad 1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27} \cdots \frac{(-1)^{n-1}}{3^{n-1}} \cdots$$

where, $r = -1/3$, that is to say $|r| < 1$, and the first term in the series $a_1 = 1$ so that

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}}{3^{n-1}} = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{3}{4}$$

1.5 Mathematical Induction

1.39. Mathematical induction technique is based on the three step procedure:

1. given identity must be true for $n = 1$, as

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \Rightarrow \quad 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \ \checkmark$$

2. given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

3. assuming that the identity is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k + 1$. That is to say,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{1.1}$$

$$\therefore \quad \{\text{add the next term } (k+1) \text{ to both sides}\}$$

$$\underbrace{1 + 2 + 3 + \cdots + k}_{\frac{k(k+1)}{2}} + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)\big((k+1)+1\big)}{2} \quad \checkmark \tag{1.2}$$

Obviously, (1.2) is in the same form as (1.1) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.1) *is true then* (1.2) *is also true for* $(k+1)$.

Final argument: for $n = 1$ (1.1) is true and, that being the case, it is true for the next $n = 2$. Consequently, being true for $n = 2$ it is also true for the next $n = 3$, etc.. By the "domino effect" (1.1) is therefore true for *all* subsequent n.

1.40. General term of this series is $a_i = i^2$, and the sum of first n squares is found to be

$$S = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1.3}$$

Proof and derivation of sum (1.3) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.3) must be true.

$$S_1 = 1^2 = \frac{1 \times (1+1)(2 \times 1+1)}{6} = \frac{1 \times 2 \times 3}{6} = \frac{6}{6} = 1 \quad \checkmark$$

which simply proves that (1.3) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k = \frac{k(k+1)(2k+1)}{6} \tag{1.4}$$

3. *Case:* $n = k+1$ assuming that the identity (1.4) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k+1$. That is to say, when the following term is added as

$$\begin{aligned} S_{k+1} &= \underbrace{1^2 + 2^2 + \cdots + k^2}_{S_k} + (k+1)^2 = S_k + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = (k+1)\frac{k(2k+1) + 6(k+1)}{6} \\ &= (k+1)\frac{2k^2 + 7k + 6}{6} = (k+1)\frac{2k^2 + 4k + 3k + 6}{6} = \frac{(k+1)\,(k+2)(2k+3)}{6} \\ &= \frac{(k+1)\,\big((k+1)+1\big)\,\big(2\,(k+1)+1\big)}{6} \end{aligned} \tag{1.5}$$

Obviously, (1.5) is in the same form as (1.4) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.4) *is true then* (1.5) *is also true for* $(k+1)$.

Final argument: for $n = 1$ (1.3) is true, and in the previous step it is proven that the next sum of $n = 2$ elements is also true. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

1.41. Given sum of n cubes,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{1.6}$$

Proof and derivation of sum (1.6) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.6) must be true.

$$S_1 = 1^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{1(1+1)}{2}\right)^2 = \frac{4}{4} = 1 \quad \checkmark$$

   which simply proves that (1.6) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k = \left(\frac{k(k+1)}{2}\right)^2 \tag{1.7}$$

3. *Case:* $n = k + 1$ assuming that the identity (1.7) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k + 1$. That is to say, when the following term is added as

$$\begin{aligned} S_{k+1} &= \underbrace{1^3 + 2^3 + 3^3 + \cdots + k^3}_{S_k} + (k+1)^3 = S_k + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 2^2(k+1)^3}{2^2} = (k+1)^2\,\frac{k^2 + 2^2(k+1)}{2^2} = (k+1)^2\,\frac{k^2 + 4k + 4}{2^2} \\ &= \left(\frac{(k+1)\,(k+2)}{2}\right)^2 = \left(\frac{(k+1)\,\big((k+1)+1\big)}{2}\right)^2 \quad \checkmark \end{aligned} \tag{1.8}$$

   Obviously, (1.7) is in the same form as (1.8) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.7) *is true then* (1.8) *is also true for* $(k+1)$.

Final argument: for $n = 1$ (1.6) is true, and in the previous step it is proven that the *next sum* of $n = 2$ elements is also true. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

1.42. Given sum of n powers of two,

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 \tag{1.9}$$

Proof and derivation of sum (1.9) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.9) must be true.

$$S_1 = 2^{1-1} = 2^1 - 1 \quad \therefore \quad 2^0 = 1 \quad \checkmark$$

which simply proves that (1.9) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k = 2^k - 1 \tag{1.10}$$

3. *Case:* $n = k + 1$ assuming that the identity (1.10) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k + 1$. That is to say, when the following term is added as

$$S_{k+1} = \underbrace{1 + 2 + 2^2 + \cdots + 2^{k-1}}_{S_k} + 2^k = S_k + 2^k = \left(2^k - 1\right) + 2^k$$

$$= 2^k + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1 \quad \checkmark \tag{1.11}$$

Obviously, (1.10) is in the same form as (1.11) after '$k \rightarrow (k + 1)$' replacement. In conclusion, *if* (1.10) *is true then* (1.11) *is also true for* $(k + 1)$.

Final argument: for $n = 1$ (1.9) is true, and in the previous step it is proven that the *next* sum, i.e. of $n = 2$ elements, is also correct. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

1.43. Given sum of n odd numbers

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{1.12}$$

Proof and derivation of sum (1.12) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.12) must be true.

$$S_1 = (2 \cdot 1 - 1) = 1^2 \quad \therefore \quad 1 = 1 \quad \checkmark$$

which simply proves that (1.12) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k = 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{1.13}$$

3. *Case:* $n = k + 1$ assuming that the identity (1.13) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k + 1$. That is to say, when the following term is added as

$$S_{k+1} = \underbrace{1+3+5+\cdots+(2k-1)}_{S_k} + (2k+1) = S_k + (2k+1) = k^2 + (2k+1)$$

$$= (k+1)^2 \quad \checkmark \tag{1.14}$$

Note that given odd number $(2k-1)$ the next odd number is $(2k-1)+2 = (2k+1)$ because all odd numbers (as well as even) are separated by ± 2. Obviously, (1.13) is in the same form as (1.14) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.13) *is true then* (1.14) *is also true for* $(k+1)$.

Final argument: for $n = 1$ (1.12) is true, and in the previous step it is proven that the *next* sum, i.e. of $n = 2$ elements, is also correct. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

1.44. Given sum of n binary products

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{1.15}$$

Proof and derivation of sum (1.15) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.15) must be true.

$$S_1 = 1\,(1+1) = \frac{1(1+1)(1+2)}{3} \quad \therefore \quad 2 = \frac{6}{3} \quad \checkmark$$

which simply proves that (1.15) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3} \tag{1.16}$$

3. *Case:* $n = k+1$ assuming that the identity (1.16) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k+1$. That is to say, when the following term is added as

$$S_{k+1} = \underbrace{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1)}_{S_k} + (k+1)(k+2) = S_k + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)\big((k+1)+1\big)\big((k+1)+2\big)}{3} \quad \checkmark \tag{1.17}$$

Obviously, (1.16) is in the same form as (1.17) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.16) *is true then* (1.17) *is also true for* $(k+1)$.

Final argument: for $n = 1$ (1.15) is true, and in the previous step it is proven that the *next* sum, i.e. of $n = 2$ elements, is also correct. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

1.45. Given sum of n inverse binary odd number products

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \tag{1.18}$$

Proof and derivation of sum (1.18) is based on the three step procedure mathematical induction technique

1. *Case:* $n = 1$ given identity (1.18) must be true.

$$S_1=\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{2\cdot 1+1}\quad\therefore\quad\frac{1}{3}=\frac{1}{3}\ \checkmark$$

   which simply proves that (1.18) is true for $n = 1$.

2. *Assumption:* given that the identity is true for $n = 1$, it should be true for other $n = k$, i.e.

$$S_k=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \tag{1.19}$$

3. *Case:* $n = k + 1$ assuming that the identity (1.19) is true for $n = k$, it must be proven that it is true for the next natural number, i.e. $n = k + 1$. That is to say, when the following term is added as

$$\begin{aligned}S_{k+1}&=\underbrace{\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}}_{S_k}+\frac{1}{(2k+1)(2k+3)}\\&=S_k+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k(2k+1)(2k+3)+(2k+1)}{(2k+1)(2k+1)(2k+3)}\\&=\frac{\cancel{(2k+1)}\big(k(2k+3)+1\big)}{\cancel{(2k+1)}(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(k+1)\cancel{(2k+1)}}{\cancel{(2k+1)}(2k+3)}=\frac{k+1}{2\,(k+1)+1}\ \checkmark\end{aligned} \tag{1.20}$$

   Obviously, (1.19) is in the same form as (1.20) after '$k \to (k+1)$' replacement. In conclusion, *if* (1.19) *is true then* (1.20) *is also true for* $(k + 1)$.

Final argument: for $n = 1$ (1.18) is true, and in the previous step it is proven that the *next* sum, i.e. of $n = 2$ elements, is also correct. Then, by induction, it must be that the next sum of $n = 3$ elements is also correct, etc., for any subsequent sum of n elements.

Check for updates

Elementary Special Functions

2

Delta function is defined as

$$\delta(t) = 0; \quad (t \neq 0)$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Geometrical interpretation of Dirac function $\delta(t)$ is that its *area* is normalized to one (as definite integral), even though its height tends to infinity while at the same time its length tends to zero (thus, strictly speaking, its area $A = 0 \cdot \infty$ is not determined).

The main properties of Dirac delta function $\delta(t)$ are the following:

1. It is an even function, i.e., $\delta(t) = \delta(-t)$
2. Sampling property, $\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0)$

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$$

$$\int_{a}^{b} x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0), & t_0 \in (a, b) \\ 0, \text{ otherwise} \end{cases}$$

Problems

2.1 Basic Special Functions

Sketch graph and give definition of special functions in P.2.1 to P.2.9.

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_2

2.1. $e^{(\sigma+j\omega)t}$, $(\sigma, \omega \in \mathbb{R})$

2.2. $\operatorname{sinc}(t)$

2.3. Heaviside $u(t)$.

2.4. $\operatorname{sign}(t)$

2.5. $r(t)$

2.6. $\Pi(t)$

2.7. $\Lambda(t)$

2.8. $\delta(t)$

2.9. $\text{Ш}_T(t)$

2.2 Derivatives and Integrals with Special Functions

Calculate derivatives and integrals in P.2.10 to P.2.29.

2.10. $u'(t)$

2.11. $\operatorname{sign}'(t)$

2.12. $r'(t)$

2.13. $\Pi'(t/2)$

2.14. $\Lambda'(t)$

2.15. $\displaystyle\int_{-t}^{t} \operatorname{sign}(\tau)\, d\tau$

2.16. $\displaystyle\int_{-\infty}^{t} u(\tau)\, dt$

2.17. $\displaystyle\int_{-\infty}^{t} r(\tau)\, dt$

2.18. $\displaystyle\int_{-\infty}^{\infty} \Pi(t)\, dt$

2.19. $\displaystyle\int_{-\infty}^{1/4} \Pi(t)\, dt$

2.20. $\displaystyle\int_{-\infty}^{\infty} \Lambda(t)\, dt$

2.21. $\displaystyle\int_{-\infty}^{1/4} \Lambda(t)\, dt$

2.22. $\displaystyle\int_{-\infty}^{\infty} \delta(t-4)\, dt$

2.23. $\displaystyle\int_{-\infty}^{3} \delta(t-4)\, dt$

2.24. $\displaystyle\int_{-\infty}^{\infty} t^2\, \delta(t+2)\, dt$

2.25. $\displaystyle\int_{-3/2}^{\infty} t^2\, \delta(t+2)\, dt$

2.26. $\displaystyle\int_{-\infty}^{\infty} \sin(2.5\pi\, t)\, \delta(t-1)\, dt$

2.27. $\displaystyle\int_{-\infty}^{\infty} e^{-at^2}\, \delta(t+1)\, dt, \quad (a > 0)$

2.28. $\displaystyle\int_{0}^{\infty} e^{-at^2}\, \delta(t+1)\, dt, \quad (a > 0)$

2.29. $\displaystyle\int_{-\infty}^{\infty} x(t)\, \delta(t-a)\, dt, \quad a \in \Re$

2.3 Special Functions Relations

Derive relationship of functions in P.2.30 to P.2.33.

2.30. $u(t)$ as function of sign (t)

2.31. $\Pi(t/2\tau)$ as function of $\delta(t)$

2.32. $\Pi(t/2\tau)$ as function of $u(t/2\tau)$

2.33. $\Pi(t)$ as function of sign (t)

Answers

2.1 Basic Special Functions

2.1. Complex exponential function

$$x(t) = e^{s\,t} \quad \text{where,} \quad s = \sigma + j\omega \quad \text{and} \quad \sigma, \omega \in \mathbb{R}$$

may be written as

$$x(t) = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}\Big(\underbrace{\cos(\omega\, t)}_{\Re(x(t))} + \; j\; \underbrace{\sin(\omega\, t)}_{\Im(x(t))}\Big)$$

$$x^*(t) = e^{(\sigma - j\omega)t} = e^{\sigma t} e^{-j\omega t} = e^{\sigma t}\big(\cos(\omega\, t) - j\, \sin(\omega\, t)\big)$$

where, $e^{\sigma t}$ is a real exponential function, while $e^{\pm j\omega t}$ is a strictly complex exponential function. Then, after applying Euler's formula, it is evident that complex exponential term by itself consists of cosine (along real axis) and sine (along the perpendicular imaginary axis) functions, Fig. 2.1 (left). As the argument t progresses, amplitudes of the two trigonometric functions is multiplied by the real exponential term.

Case 1: $\sigma = 0, \omega = 0$: complex exponential function becomes a constant.

$$x(t) = e^{(\sigma + j\omega)t} = e^{(0 + j\,0)t} = e^0 = 1 \quad \{\text{a real constant}\}$$

Case 2: $\sigma > 0, \omega = 0$: real part is positive and imaginary part is zero. Therefore, only the real exponential term is left.

$$x(t) = e^{(\sigma + j\omega)t} = e^{(\sigma + j\,0)t} = e^{\sigma\, t} \quad \{\text{a real exponential function}\}$$

Case 3: $\sigma = 0, \omega \neq 0$: real part is zero and imaginary part is non–zero, therefore only the strictly complex exponential term is left. By consequence, amplitudes of both sine and cosine functions are constant ± 1, Fig. 2.1 (right).

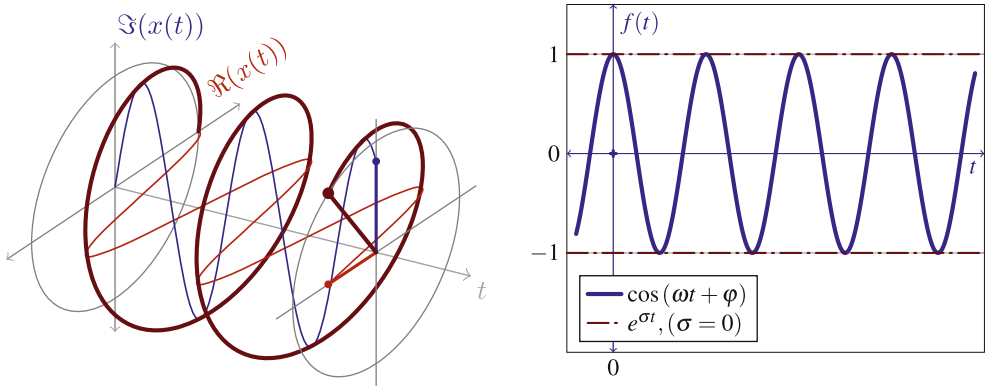


Fig. 2.1 Example P.2.1

$$x(t) = e^{(\sigma+j\omega)t} = e^{(0+j\omega)t} = e^{j\omega t} = \big(\cos(\omega t) + j\,\sin(\omega t)\big)$$

$$\{\text{a strictly complex exponential function}\}$$

In addition, the sum and difference of complex conjugate exponents give

$$\begin{aligned} x(t) + x^*(t) &= e^{j\omega t} + e^{-j\omega t} \\ &= \big(\cos(\omega t) + j\,\sin(\omega t)\big) + \big(\cos(\omega t) - j\,\sin(\omega t)\big) \\ &= 2\cos(\omega t) \qquad \therefore \\ \cos(\omega t) &= \frac{e^{j\omega t} + e^{-j\omega t}}{2} \end{aligned}$$

$$\begin{aligned} x(t) - x^*(t) &= e^{j\omega t} - e^{-j\omega t} \\ &= \big(\cos(\omega t) + j\,\sin(\omega t)\big) - \big(\cos(\omega t) - j\,\sin(\omega t)\big) \\ &= 2j\,\sin(\omega t) \qquad \therefore \\ \sin(\omega t) &= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \end{aligned}$$

Case 4: $\sigma > 0, \omega \neq 0$: increasing real exponential function multiplies amplitude of both cosine function along the real axis, and sine function along the imaginary axis, Fig. 2.2 (left).

Case 5: $\sigma < 0, \omega \neq 0$: decreasing real exponential function multiplies amplitude of both cosine function along the real axis, and sine function along the imaginary axis, Fig. 2.2 (right).

Reminder: Depending on specific values of $(\sigma, \omega \in \mathbb{R})$ complex exponential function $e^{(\sigma+j\omega)t}$ transforms to a constant or other fundamental engineering functions.

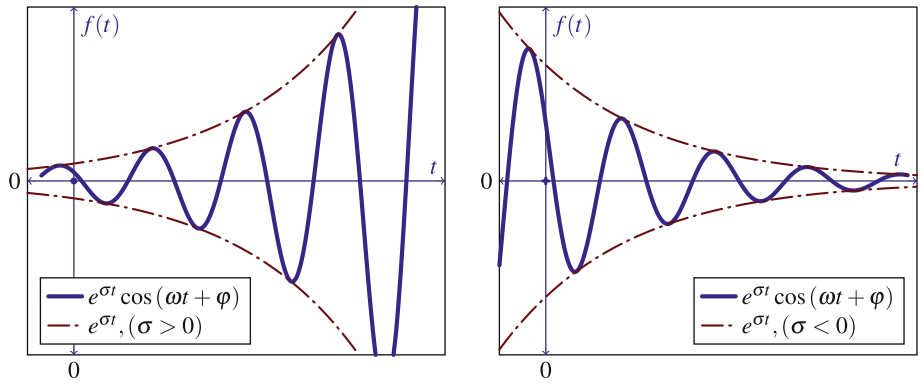


Fig. 2.2 Example P.2.1

Fig. 2.3 Example P.2.2

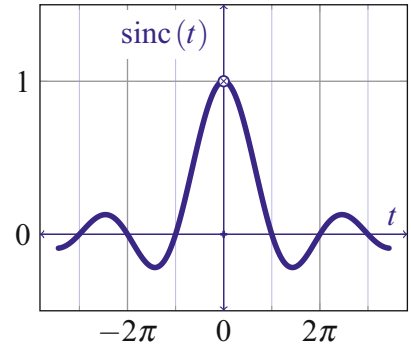
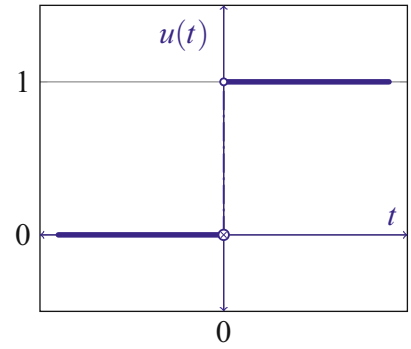


Fig. 2.4 Example P.2.3



2.2. Cardinal sine function sinc(t), see Fig. 2.3, is analytically defined as

$$\text{sinc}\,(t) = \frac{\sin t}{t} \quad \text{or,} \quad \text{sinc}\,(\pi\, t) = \frac{\sin(\pi\, t)}{\pi\, t}$$

where the last definition is normalized to π. Cardinal sinus function is *not* defined for $t = 0$, however when $t \to 0$ it limits as sinc $(0) \to 1$ (see the limits chapter in Calculus book). To indicate that point $(0, 1)$ is *not* included, it is marked with the crossed circle symbol.

2.3. Heaviside $u(t)$, see Fig. 2.4, is a piecewise linear function that is analytically defined as

$$u(t) = \begin{cases} 0, & t < 0 \quad \text{i.e.} \quad \lim_{t \to \underset{\to}{0}} u(t) = 0 \\ 1, & t \geq 0 \quad \text{i.e.} \quad \lim_{t \to \underset{\leftarrow}{0}} u(t) = 1 \end{cases}$$

Step function $u(t)$ is not continuous at $t = 0$, where by agreement $u(t) = 1$ for $t \geq 0$. To indicate that point $(0, 0)$ is *not* defined, it is marked with the crossed circle symbol. Note that limit at $t = 0$ does not exist because the left and right side limits are different.

2.4. Piecewise linear sign(t) function, see Fig. 2.5, is analytically defined as,

Fig. 2.5 Example P.2.4

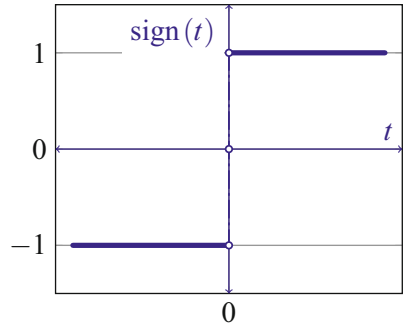
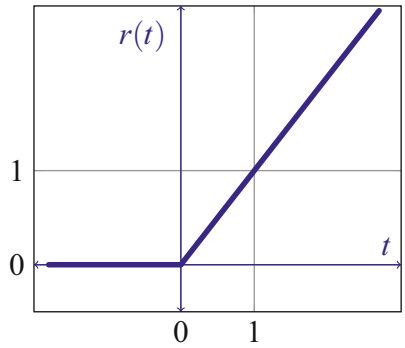


Fig. 2.6 Example P.2.5



$$\operatorname{sign}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \quad \text{i.e.} \quad \lim_{t \to \underset{\rightarrow}{0}} \operatorname{sign}(t) = -1 \\ 1, & t > 0 \quad \text{i.e.} \quad \lim_{t \to \underset{\leftarrow}{0}} \operatorname{sign}(t) = 1 \end{cases}$$

or, alternatively, relative to absolute function

$$\operatorname{sign}(t) = \frac{t}{|t|} = \frac{|t|}{t}, \quad \forall x \neq 0$$

By agreement, point of discontinuity at $t = 0$ is declared as $\operatorname{sign}(0) = 0$.

2.5. Piecewise linear ramp function $r(t)$, see Fig. 2.6, is analytically defined as follows.

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

Note that ramp function is well defined for $t = 0$, however the left and right side *derivatives* are not the same ($r'(t) = 0, \ t \leq 0$, and $r'(t) = 1, \ t \geq 0$). Alternative analytical definition of ramp function may be

$$r(t) = t\, u(t)$$

because, for $t < 0$ the product of any function $\big($thus including linear function $y(t) = t\big)$ with $u(t)$ equal zero.

Fig. 2.7 Example P.2.6

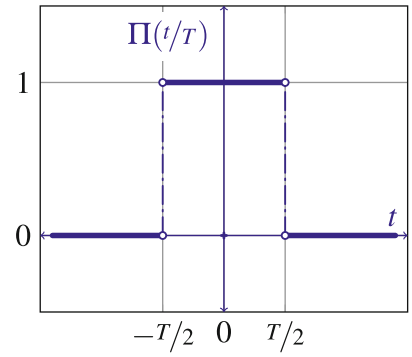
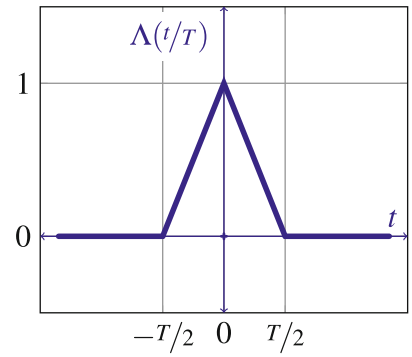


Fig. 2.8 Example P.2.7



2.6. Piecewise rectangular function $\Pi(t)$, Fig. 2.7, is analytically defined as

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} \geq t \geq \frac{T}{2} \\ 0, \text{ otherwise} \end{cases}$$

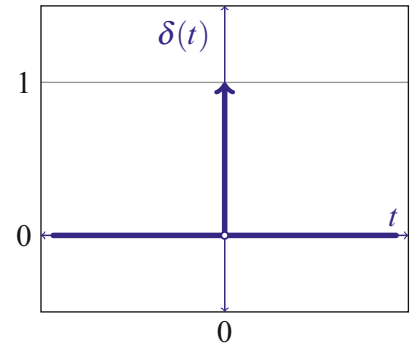
Where, T is the length of non–zero interval centred around the origin. By agreement, the two points of discontinuities at $t = \pm T/2$ are declared as $\Pi(\pm T/2) = 1$. It should be evident that limits at two discontinuity points $t = \pm T/2$ are not existent. As shown in Ch. 5, P.5.10 and P.5.15, $\Pi(t)$ and $\text{sinc}(\omega)$ are dual functions in respect to Fourier transformation.

2.7. Triangle function $\Lambda(t)$, Fig. 2.8, is analytically defined as

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \frac{t}{T/2} + 1, & -\frac{T}{2} \leq t \leq 0 \\ 1 - \frac{t}{T/2}, & 0 \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Where, parameter T is the length of non–zero interval centred around the origin. Note that triangle function is well defined for $\forall t$, however the left and right side *derivatives* at $t = \pm T/2$ and $t = 0$ are not the same.

Fig. 2.9 Example P.2.8



Note that $\Lambda'(t) = 0,\ |t| \geq \pm T/2$, but $\Lambda'(t) = 2/T,\ -T/2 \leq t \leq 0$, $\Lambda'(t) = -2/T,\ 0 \leq t \leq T/2$. As shown in Ch. 5, P.5.11, $\Lambda(t)$ and $\text{sinc}^2(\omega)$ are dual functions in respect to Fourier transformation.

2.8. Dirac delta function $\delta(t)$, Fig. 2.9, is defined by the means of definite integral as

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(t) = 0, \quad t \neq 0$$

Dirac delta function therefore equal zero for all t, *except* at $t = 0$ where, because being defined by definite integral, its *area* is normalized to one, as indicated at y–axis (its height is infinite). By consequence, the *product* of any continuous function and Dirac delta results in *only one* non–zero point, assuming that Dirac delta is found in the given interval.

Reminder: Main properties of $\delta(t)$ are:

1. it is an even function, i.e. $\delta(t) = \delta(-t)$
2. sampling property, $\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0)$

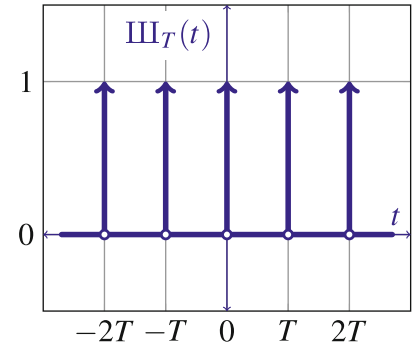
$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$$

$$\int_{a}^{b} x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0),\ t_0 \in (a, b) \\ 0, \text{ otherwise} \end{cases}$$

2.9. Dirac comb (a.k.a. "sampling") periodic function $\text{Ш}_T(t)$ is created by a sum of equidistant Dirac functions, Fig. 2.10, analytically defined as

$$\text{Ш}_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Fig. 2.10 Example P.2.9



where, T is sampling period, thus sampling frequency is $f_s = 1/T$. Using periodically placed Dirac delta functions enables sampling of arbitrary number of sampling points—each product of the particular Delta function with any continuous function results in one non–zero product. Dirac comb is the key function for "analog to digital" (A/D) signal conversion.

2.2 Derivatives and Integrals with Special Functions

2.10. Derivative $u'(t)$ may be deduced from graphical interpretation, Fig. 2.11 (left). Obviously in the two intervals where $t \neq 0$ the step function is constant, therefore by definition, its derivative is

1. *Case* $t < 0$: then $u(t) = 0 \Rightarrow u'(t) = 0$,
2. *Case* $t > 0$: then $u(t) = 1 \Rightarrow u'(t) = 0$, and
3. *Case* $t = 0$: then as the interval between $u(t) = 0$ and $u(t) = 1$ tends to zero, i.e. $\Delta t \to 0$, *the rate of change* between '0' and '1' tends to infinity, which by definition creates Dirac function. In other words, as $t \to 0$,

$$\frac{d\,u(t)}{dt} = \delta(t)$$

In summary, the zero time change between between '0' and '1' levels shows as Delta function. Physical interpretation may be that zero time change requires an infinite amount of energy.

2.11. Derivative $\text{sign}'(t)$ may be deduced from graphical interpretation, Fig. 2.11 (right). In the two intervals where $t \neq 0$ sign function is constant, by definition, its derivative is

1. *Case* $t > 0$: then $\text{sign}(t) = +1 \Rightarrow \text{sign}'(t) = 0$,
2. *Case* $t < 0$: then $\text{sign}(t) = -1 \Rightarrow \text{sign}'(t) = 0$,
3. *Case* $t = 0$: then as the interval between $\text{sign}(t) = -1$ and $\text{sign}(t) = 1$ tends to zero, i.e. $\Delta t \to 0$, *the rate of change* between '-1' and '1' tends to infinity, which by definition describes Dirac function. However, the vertical change covers two units, each associated with "one Dirac", in other words, as $t \to 0$,

$$\frac{d\,\text{sign}(t)}{dt} = 2\delta(t)$$

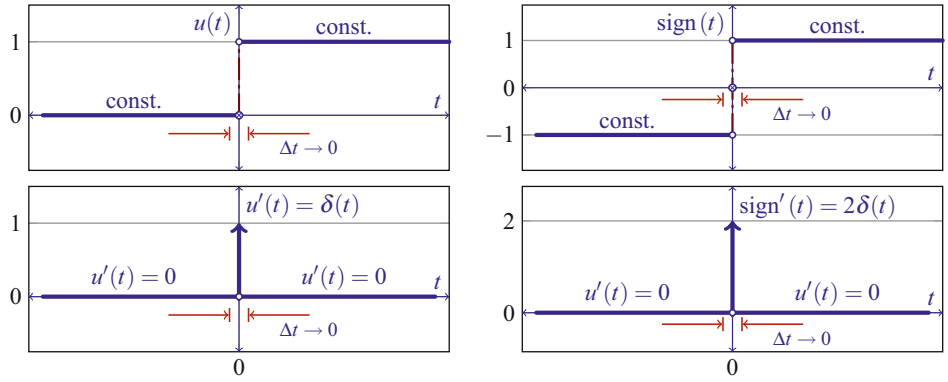
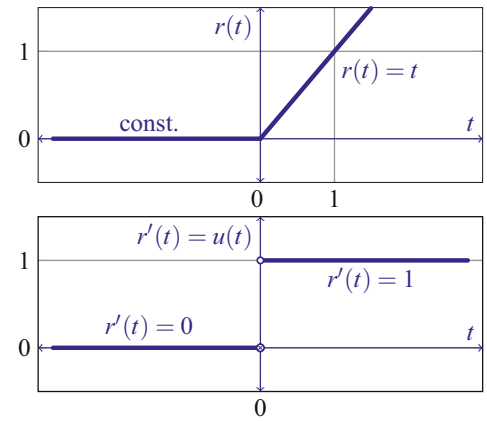


Fig. 2.11 Examples P.2.10, P.2.11

Fig. 2.12 Example P.2.12



In summary, the zero time change between between '±1' levels shows as two times Delta function.

2.12. Derivative $r'(t)$ may be deduced from graphical interpretation, Fig. 2.12. Obviously for $t < 0$ ramp function is constant and it is linear for $t \geq 0$, therefore by definition its derivative is

1. *Case* $t < 0$: then $r(t) = 0 \Rightarrow r'(t) = 0$,
2. *Case* $t \geq 0$: then $r(t) = t \Rightarrow r'(t) = 1$

In conclusion, for $t < 0$ derivative $r'(t) = 0$ and for $t \geq 0$ derivative $r'(t) = 1$, which is identical to definition of Heaviside step function $u(t)$.

2.13. Derivative $\Pi'(t/2)$ may be deduced from graphical interpretation, Fig. 2.13. Obviously, except at the two discontinuity points $t = \pm 1$ rectangular function is constant, therefore by definition, its derivative is

1. *Case* $t < -1$: then $\Pi(t/2) = 0 \Rightarrow \Pi'(t/2) = 0$,
2. *Case* $-1 \leq t \leq 1$: then $\Pi(t/2) = 1 \Rightarrow \Pi'(t/2) = 0$,

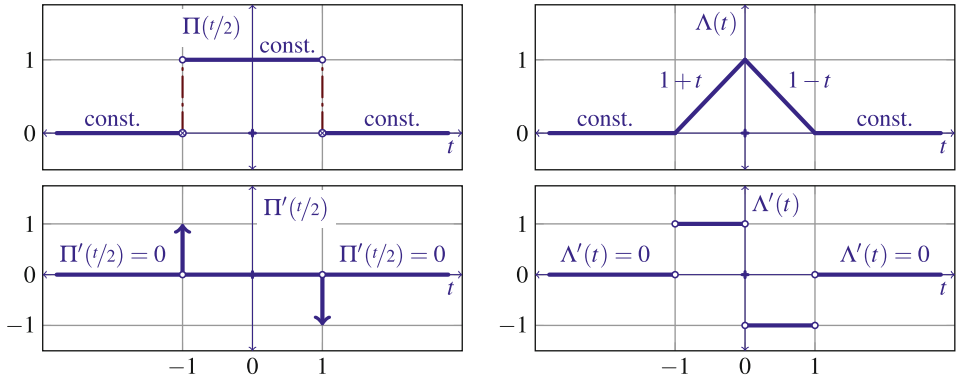


Fig. 2.13 Examples P.2.13, P.2.14

3. *Case* $t > 1$: then $\Pi(t/2) = 0 \Rightarrow \Pi'(t/2) = 0$,
4. *Case* $t = -1$: then as the interval between $\Pi(t/2) = 0$ and $\Pi(t/2) = 1$ tends to zero, i.e. $\Delta t \to 0$, *the rate of change* between '0' and '1' tends to infinity, which by definition describes Dirac function. In other words, as $t \to -1$ (see A.2.22),

$$\frac{d\,\Pi(t/2)}{dt} = \delta(t+1), \quad (t = -1)$$

5. *Case* $t = 1$: then as the interval between $\Pi(t/2) = 0$ and $\Pi(t/2) = -1$ tends to zero, i.e. $\Delta t \to 0$, *the rate of change* between '0' and '-1' tends to negative infinity, which by definition describes negative Dirac function. In other words, as $t \to 1$,

$$\frac{d\,\Pi(t/2)}{dt} = -\delta(t-1), \quad (t = 1)$$

In summary, at each discontinuity point of $\Pi(t/2)$ the rate of function's change tends to infinity; to positive infinity at $t = -1$ and negative infinity at $t = 1$, as indicated with the two Dirac functions in Fig. 2.13 (left).

2.14. Derivative $\Lambda'(t)$ may be deduced from graphical interpretation, Fig. 2.13 (right). There are four distinct intervals, thus its derivative is

1. *Case* $t < -1$: then $\Lambda(t) = 0 \Rightarrow \Lambda'(t) = 0$,
2. *Case* $-1 \le t \le 0$: then $\Lambda(t) = 1 + t \Rightarrow \Lambda'(t) = 1$,
3. *Case* $0 \le t \le 1$: then $\Lambda(t) = 1 - t \Rightarrow \Lambda'(t) = -1$,
4. *Case* $t > 1$: then $\Lambda(t) = 0 \Rightarrow \Lambda'(t) = 0$,

In summary, even though $\Lambda(t)$ is continuous, its derivative is a piecewise linear non–continuous function.

2.15. Sign function equal '-1' for negative t, and '$+1$' for $t \ge 0$, thus (by employing temporarily variable τ)

Case 1: $\tau \leq 0$

$$\int_{-t}^{0} \text{sign}\,(\tau)\ d\tau = \int_{-t}^{0} \text{sign}\,(\tau)\ d\tau = \int_{-t}^{0} (-1)\, d\tau = -\tau \Big|_{-t}^{0} = -\big(0-(-t)\big) = -t$$

Case 2: $\tau \geq 0$

$$\int_{0}^{t} \text{sign}\,(\tau)\ d\tau = \int_{0}^{t} \text{sign}\,(\tau)\ d\tau = \int_{0}^{t} (1)\, d\tau = \tau \Big|_{0}^{t} = (t-0) = t$$

In summary, see Fig. 2.14

$$\int \text{sign}\,(t)\ dt = \begin{cases} -t & (t<0) \\ t & (t \geq 0) \end{cases} \overset{\text{def}}{=} |t|$$

2.16. Step function equal zero for $t < 0$, and equal one (i.e. a constant) for $t \geq 0$, thus (by employing temporarily variable τ)

Case 1: $\tau \leq 0$

$$\int_{-\infty}^{0} u(\tau)\, d\tau = \int_{-\infty}^{0} u(\tau)\, d\tau = \int_{-\infty}^{0} 0\, d\tau = 0$$

Case 2: $\tau \geq 0$

$$\int_{0}^{t} u(\tau)\, d\tau = \int_{0}^{t} u(\tau)\, d\tau = \int_{0}^{t} 1\, d\tau = \int_{0}^{t} d\tau = \tau \Big|_{0}^{t} = t$$

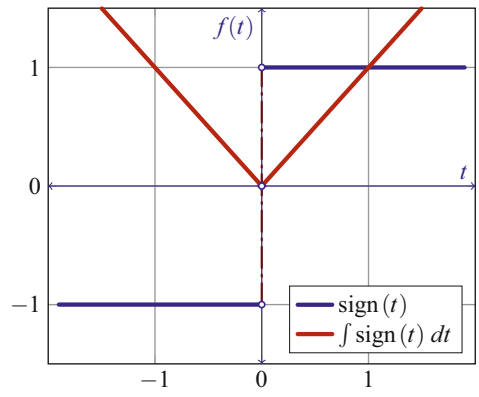


Fig. 2.14 Example P.2.15

In summary,

$$\int_{-\infty}^{t} u(\tau)\, d\tau = \begin{cases} t; & t \geq 0 \\ 0; & t < 0 \end{cases} = r(t)$$

which should be easy to visualize, for $t \geq 0$, as area of a rectangle whose height equal one and its length, i.e. t, increases linearly increases.

2.17. Ramp function equal zero for $t < 0$, and t for $t \geq 0$, thus (by using temporarily variable τ)

Case 1: $\tau \leq 0$

$$\int_{-\infty}^{0} r(\tau)\, d\tau = \int_{-\infty}^{0} r(\tau)\, d\tau = \int_{-\infty}^{0} 0\, d\tau = 0$$

Case 2: $\tau \geq 0$

$$\int_{0}^{t} r(\tau)\, d\tau = \int_{0}^{t} r(\tau)\, d\tau = \int_{0}^{t} \tau\, d\tau = \frac{1}{2}\, \tau^2 \Big|_{0}^{t} = \frac{t^2}{2}$$

In conclusion,

$$\int_{-\infty}^{t} r(\tau)\, d\tau = \begin{cases} \dfrac{t^2}{2}; & t \geq 0 \\ 0; & t < 0 \end{cases} = \frac{t^2}{2}\, u(t)$$

2.18. By definition, see P.2.6, rectangular function is

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & -\dfrac{T}{2} \geq t \geq \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

that is to say, $T = 1$, see Fig. 2.15 (left). Defined integral results in area between function and the horizontal axis, in this case calculated over three distinct intervals

$$\int_{-\infty}^{\infty} \Pi(t)\, dt = \int_{-\infty}^{-1/2} \Pi(t)\, dt + \int_{-1/2}^{1/2} \Pi(t)\, dt + \int_{1/2}^{\infty} \Pi(t)\, dt = \int_{-\infty}^{-1/2} 0\, dt + \int_{-1/2}^{1/2} 1\, dt + \int_{1/2}^{\infty} 0\, dt$$
$$= \int_{-1/2}^{1/2} dt = t \Big|_{-1/2}^{1/2} = 1$$

The only non–zero area is within the $(-1/2, 1/2)$ interval. From the elementary geometry, it is evident that rectangular area is height times length, which in this case equal one square unit despite infinite bounds of the integral (because, elsewhere port function equal zero).

2.19. By definition, see P.2.6, rectangular function is

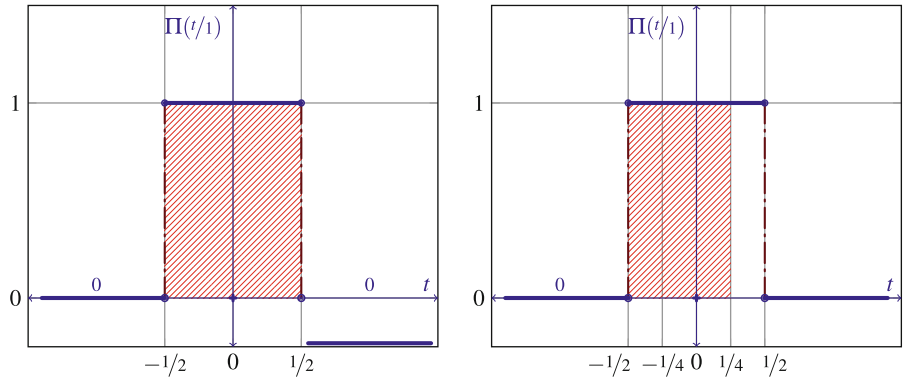


Fig. 2.15 Examples P.2.18, P.2.19

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & -\dfrac{T}{2} \geq t \geq \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

i.e. $T = 1$, see Fig. 2.15 (right). Resulting non–zero area between function and horizontal axis, is

$$\int_{-\infty}^{\infty} \Pi(t)\, dt = \int_{-\infty}^{-1/2} \Pi(t)\, dt + \int_{-1/2}^{1/4} \Pi(t)\, dt = \int_{-\infty}^{-1/2} 0\, dt + \int_{-1/2}^{1/4} 1\, dt$$
$$= \int_{-1/2}^{1/4} dt = t\,\Big|_{-1/2}^{1/4} = \frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{3}{4}$$

Evidently rectangular area is height times length, i.e. three quarters of a square unit.

2.20. By definition, see P.2.7, triangular function is

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \dfrac{t}{T/2} + 1, & -T/2 \leq t \leq 0 \\ 1 - \dfrac{t}{T/2}, & 0 \leq t \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

that is to say, $T = 1$, see Fig. 2.16 (left). That being the case, this triangular function is

$$\Lambda\,(t) = \begin{cases} 2t + 1, & -1/2 \leq t \leq 0 \\ 1 - 2t, & 0 \leq t \leq 1/2 \\ 0, \text{otherwise} \end{cases}$$

Result of a defined integral is area between function and horizontal axis, as

$$\int_{-\infty}^{\infty} \Lambda(t)\, dt = \int_{-\infty}^{-1/2} \Lambda(t)\, dt + \int_{-1/2}^{0} \Lambda(t)\, dt + \int_{0}^{1/2} \Lambda(t)\, dt + \int_{1/2}^{\infty} \Lambda(t)\, dt$$

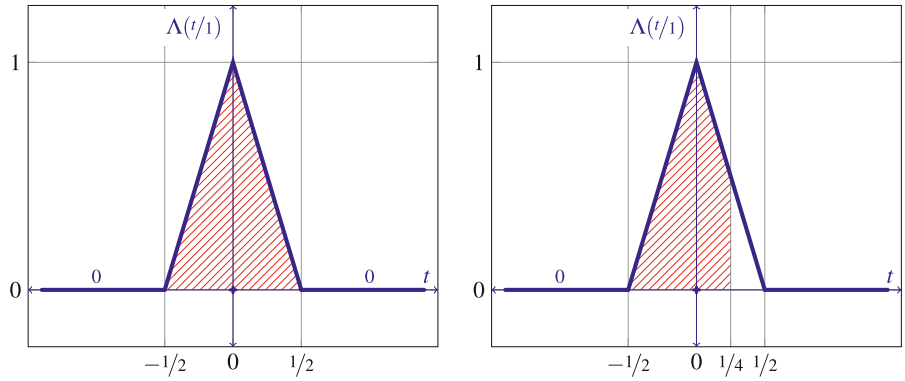


Fig. 2.16 Examples P.2.20, P.2.21

$$= \int_{-\infty}^{-1/2} 0\, dt + \int_{-1/2}^{0} (1+2t)\, dt + \int_{0}^{1/2} (1-2t)\, dt + \int_{1/2}^{\infty} 0\, dt$$

$$= t\Big|_{-1/2}^{0} + t^2\Big|_{-1/2}^{0} + t\Big|_{0}^{1/2} - t^2\Big|_{0}^{1/2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$$

Evidently, triangular area is height times length divided by two, i.e. a half square unit.

2.21. By definition, see P.2.7, triangular function is

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \dfrac{t}{T/2} + 1, & -\dfrac{T}{2} \le t \le 0 \\ 1 - \dfrac{t}{T/2}, & 0 \le t \le \dfrac{T}{2} \\ 0, \text{ otherwise} \end{cases}$$

that is to say, $T = 1$, see Fig. 2.16 (right). That being the case, this triangular function is

$$\Lambda\,(t) = \begin{cases} 2t + 1, & -1/2 \le t \le 0 \\ 1 - 2t, & 0 \le t \le 1/2 \\ 0, \text{ otherwise} \end{cases}$$

Result of a defined integral is area between function and horizontal axis, in this case calculated over three distinct intervals

$$\int_{-\infty}^{\infty} \Lambda(t)\, dt = \int_{-\infty}^{-1/2} \Lambda(t)\, dt + \int_{-1/2}^{0} \Lambda(t)\, dt + \int_{0}^{1/4} \Lambda(t)\, dt$$

$$= \int_{-\infty}^{-1/2} 0\, dt + \int_{-1/2}^{0} (1+2t)\, dt + \int_{0}^{1/4} (1-2t)\, dt$$

$$= t\Big|_{-1/2}^{0} + t^2\Big|_{-1/2}^{0} + t\Big|_{0}^{1/4} - t^2\Big|_{0}^{1/4} = \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}$$

The only non–zero area is within the $(-1/2, 1/4)$ interval, which in this case equal $7/16$ square units because of the integral bounds that do not cover complete non–zero interval.

2.22. By definition, see P.2.8, Dirac delta function is

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(t) = 0, \quad t \neq 0$$

Another way of reading this definition is to say that Dirac delta is located at coordinate where *its argument equal zero*. That is to say, $\delta(t)$ is located at $t = 0$ and equal zero for all $t \neq 0$.

So where is $\delta(t-4)$ located then? Its argument is '$t-4$', therefore this Dirac delta is located at the coordinate where *its argument equal zero*, that is

$$t - 4 = 0 \Rightarrow \underline{t = 4}$$

as illustrated in Fig. 2.17 (left). That being the case, the coordinate of Dirac delta $t = 4$ is still within the integration interval $(-\infty, \infty)$, see Fig. 2.17 (left). By consequence, the calculated integral equal zero for all $t \neq 4$ except in single point $t = 4$ that is covered by Dirac delta, thus

$$\int_{-\infty}^{\infty} \delta(t-4)\, dt = 1$$

2.23. As already shown in P.2.22, argument of this Dirac delta is '$t-4$', therefore it is located at the coordinate where *its argument equal zero*, that is

$$t - 4 = 0 \Rightarrow \underline{t = 4}$$

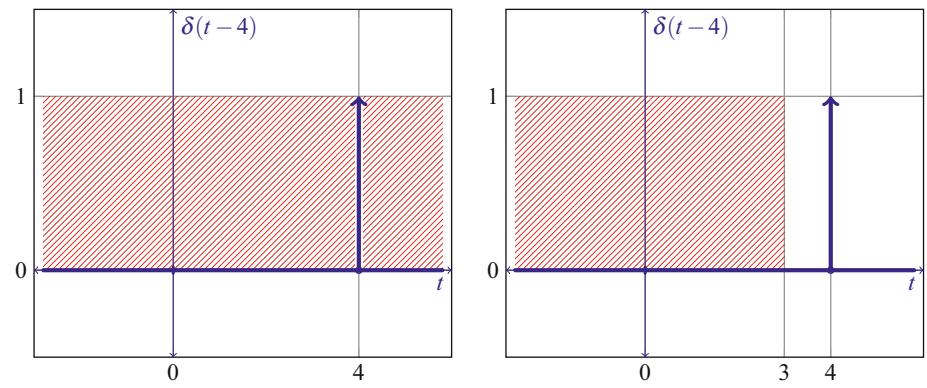


Fig. 2.17 Examples P.2.22, P.2.23

as illustrated in Fig. 2.17 (right). That being the case, the coordinate $t = 4$ is *not anymore* within the integration interval $(-\infty, 3)$, which is to say that the non–zero area of Dirac delta (equal to one) is *not* included in the total calculated area whose interval is illustrated by shading in Fig. 2.17 (right). By consequence, within given integration interval Dirac delta function always equal zero, thus

$$\int_{-\infty}^{3} \overset{0}{\cancel{\delta(t-4)}}\, dt = 0$$

2.24. As already shown in P.2.22, argument of this Dirac delta is '$t+2$', therefore it is located at the coordinate where *its argument equal zero*, that is

$$t + 2 = 0 \Rightarrow \underline{t = -2}$$

Given $f(t) = t^2$, product $f(t)\,\delta(t+2)$ equal zero within the shaded interval, see Fig. 2.18 (left), because $\delta(t+2) = 0$ for all $t \neq -2$, by consequence '$0 \cdot f(t) = 0$'. However, at single point $t = -2$ Dirac delta is not equal zero, and $f(-2) = (-2)^2 = 4$. That being the case, their non–zero product, and therefore integral, is calculated as

Case 1: $\underline{t \neq -2}$

$$\int_{-\infty}^{\infty} t^2\,\delta(t+2)\,dt = \int_{-\infty}^{\infty} t^2\,0\,dt = \int_{-\infty}^{\infty} 0\,dt = 0$$

Case 2: $\underline{t = -2}$

$$\int_{-\infty}^{\infty} t^2\,\delta(t+2)\,dt = \int_{-\infty}^{\infty} (-2)^2\,\delta(t+2)\,dt = 4\overset{1}{\cancel{\int_{-\infty}^{\infty} \delta(t+2)\,dt}} = 4$$

because, see A.2.22, Dirac delta at $\delta(t+2)$ is still within the integration interval, thus its integral equal one. This is known as "sampling property" of '$f(t)\,\delta(t)$' product, see Fig. 2.18 (left).

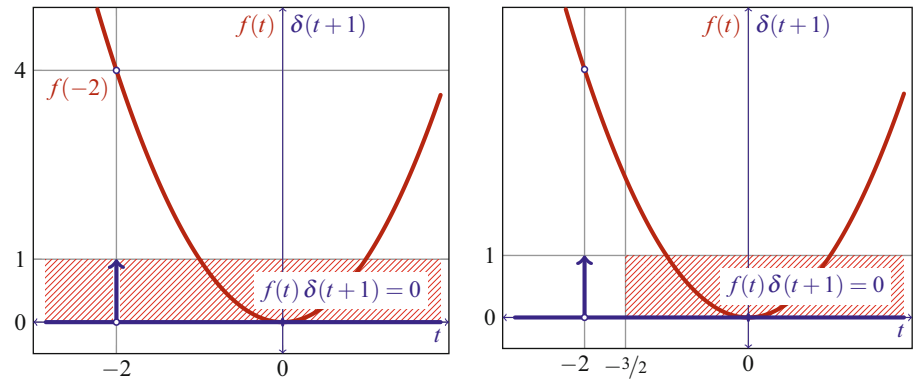


Fig. 2.18 Examples P.2.24, P.2.25

2.25. As already shown in P.2.22, argument of this Dirac delta is '$t+2$', therefore it is located at the coordinate where *its argument equal zero*, that is

$$t+2=0 \Rightarrow \underline{t=-2}$$

that is to say, *outside* of the given interval $(-3/2, +\infty)$. Given $f(t)=t^2$, product $f(t)\,\delta(t+2)$ equal zero within the integration interval (shaded), see Fig. 2.18 (right), because $\delta(t+2)=0$ for all $t \neq -2$ and by consequence '$0 \cdot f(t)=0$'. As Dirac delta is *outside* of the integration interval,

$$\int_{-3/2}^{\infty} t^2\,\delta(t+2)\,dt = \int_{-3/2}^{\infty} t^2\,0\,dt = \int_{-3/2}^{\infty} 0\,dt = 0$$

which is expected, see A.2.23, as Dirac delta at $\delta(t+2)$ is *not* within the integration interval.

2.26. Given,

$$\int_{-\infty}^{\infty} \sin(2.5\,\pi\,t)\,\delta(t-1)\,dt$$

it follows that Dirac delta is located at '$t-1=0 \Rightarrow t=1$'. Consequently, the product '$\sin(2.5\,\pi\,t)\,\delta(t-1)$' is not equal zero only at '$t=1$', see Fig. 2.19. Thus,

$$\int_{-\infty}^{\infty} \sin(2.5\,\pi\ t)\,\delta(t-1)\,dt = \int_{-\infty}^{\infty} \underbrace{\sin(2.5\,\pi \times 1)}_{\text{const.}}\,\delta(t-1)\,dt = \sin(2.5\,\pi)\int_{-\infty}^{\infty} \delta(t-1)\,dt \;\overset{1}{\nearrow}$$

$$= \sin(2.5\,\pi) = \sin(\pi/2)$$

$$= 1$$

2.27. Given

$$\int_{-\infty}^{\infty} e^{-at^2}\,\delta(t+1)\,dt, \quad (a>0)$$

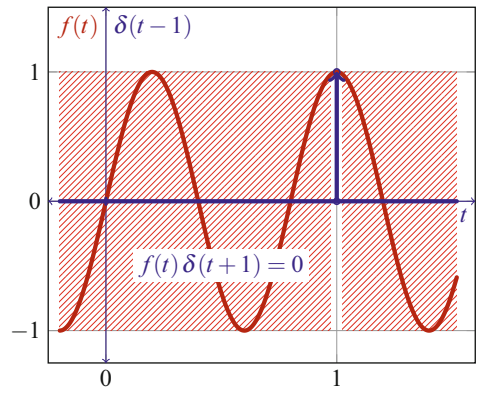


Fig. 2.19 Example P.2.26

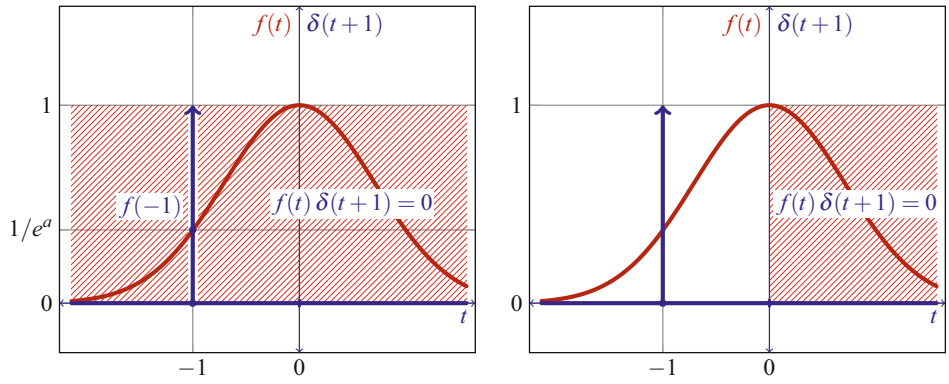


Fig. 2.20 Examples P.2.27, P.2.28

position of $\delta(t+1)$ is at $t+1=0 \;\;\therefore\;\; t=-1$, which is *inside* the bonds of given definite integral, see Fig. 2.20 (left). Then,

$$\int_{-\infty}^{\infty} e^{-at^2}\,\delta(t+1)\,dt = \int_{-\infty}^{\infty} e^{-a(-1)^2}\,\delta(t+1)\,dt = e^{-a}\int_{-\infty}^{\infty} \delta(t+1)\,dt \;\to\; 1 \;= e^{-a}$$

2.28. As oppose to P.2.27, as $\delta(t)$ is *not* found within the bonds of definite integral, then for '$t \neq -1$' all products equal zero because $\delta(t)=0$, see Fig. 2.20 (right), and,

$$\int_{0}^{\infty} e^{-at^2}\,\delta(t+1)\,dt = \int_{0}^{\infty} e^{-at^2}\,\delta(t+1) \;\to\; 0\; dt = \int_{0}^{\infty} 0\,dt = 0$$

2.29. Considering that $\delta(t-a)$ is located at $t=a$, which is certainly within interval $t \in (-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-a)\,dt = \int_{-\infty}^{\infty} x(a)\,\delta(t-a)\,dt = x(a)\int_{-\infty}^{\infty} \delta(t-a)\,dt \;\to\; 1 \;= x(a)$$

2.3 Special Functions Relations

2.30. One way to derive sign (t) as function of $u(t)$ may be by a couple of simple transformations as illustrated in Fig. 2.21. In the first step, see Fig. 2.21 (left), $u(t)$ is multiplied by factor two to create $2u(t)$ whose amplitude equal two when $t \geq 0$, see Fig. 2.21 (centre). In the second step $2u(t)$ is shifted down to create $2u(t)-1$, which is identical to sign (t) . Thus,

$$\text{sign}\,(t) = 2u(t) - 1$$

As a side note, derivative

$$\frac{d}{dt}\big(2u(t)-1\big) = 2\frac{d}{dt}u(t) - \frac{d}{dt}(1) = \{\text{see A.2.10}\} = 2\delta(t)$$

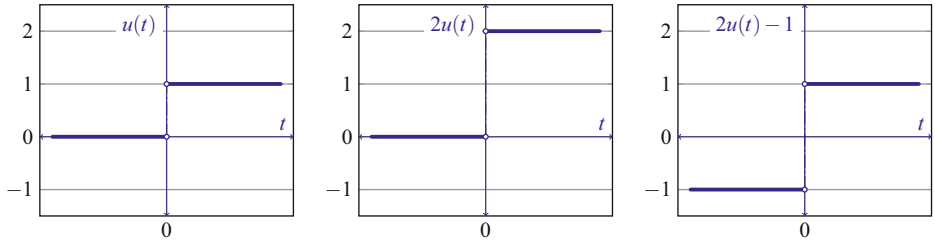
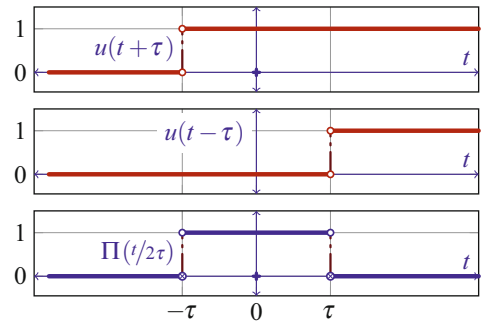


Fig. 2.21 Example P.2.30

Fig. 2.22 Example P.2.32



which is same result as already derived in A.2.11.

2.31. As already shown in P.2.13 and Fig. 2.13, derivative of rectangular function $\Pi(t/2T)$ may be expressed as function of Dirac delta as

$$\frac{d}{dt}\,\Pi(t/2T) = \delta\,(t+T) - \delta\,(t-T)$$

2.32. One possible relation between $\Pi(t/2\tau)$ and $u(t)$ is illustrated in Fig. 2.22, where one step function may be advanced and one delayed in time. The difference between these two $u(t)$ functions produces $\Pi(t/2\tau)$ as

$$\Pi\,(t/2\tau) = u\,(t+\tau) - u\,(t-\tau)$$

which is deduced by calculating the difference within each piecewise interval. For example, within the interval $t \in (-\tau, \tau)$,

$$u(t+\tau) = 1 \quad \text{and} \quad u(t-\tau) = 0$$

$$\therefore$$

$$u(t+\tau) - u(t-\tau) = 1 - 0 = 1$$

2.33. One possible relation between $\Pi(t)$ and $\text{sign}\,(t)$ is illustrated in Fig. 2.23. On the left side, it is shown how to create two $\text{sign}\,(t)$ functions: one advanced by τ, Fig. 2.23 (left, top), one

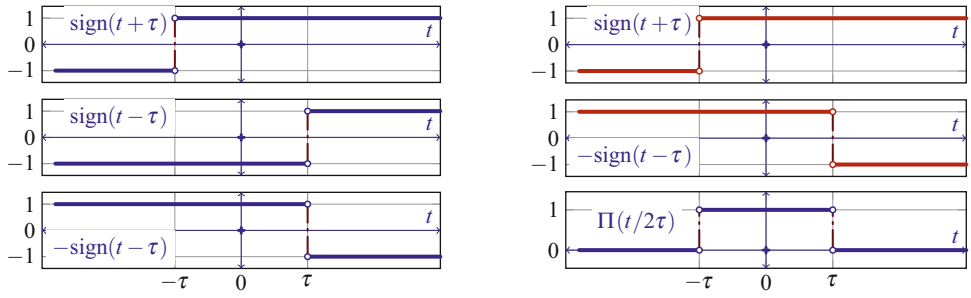


Fig. 2.23 Example P.2.33

delayed by τ, Fig. 2.23 (left, centre), and then inverted, Fig. 2.23 (left, bottom). Then, two functions sign $(t+\tau)$ and sign $(t-\tau)$ are added and divided by two (because $1+1=2$) to create $\Pi(t/2\tau)$, see Fig. 2.23 (right), as summarized by equation

$$\Pi\left(\frac{t}{2\tau}\right) = \frac{1}{2}\text{sign}\,(t+\tau) - \frac{1}{2}\text{sign}\,(t-\tau)$$

Continuous Time Convolution

3

Convolution integral is defined as

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\ d\tau \tag{3.1}$$

where τ is the provisional integration variable. That is to say, at any given instance of time t, the *products* of the two functions $x(\tau)$ and $y(t-\tau)$ are integrated relative to the parametric variable τ. It is important to realize that, being definite integral, the result of convolution integral may be interpreted as time progression of "area under the curve," where in this case "the curve" is *the product* $x(\tau)h(t-\tau)$ functions. Therefore, in the preparation part, it is necessary to perform a series of function transformations to realize the two required functions, i.e., $x(t) \rightarrow x(\tau)$ and $y(t) \rightarrow y(t-\tau)$.

Energy and power: the basic relationship between energy and power within a time interval t is

$$E(t) = \int_0^t P(\tau)\ d\tau$$

Given a signal $x(t)$, its energy E_x and average power $\langle P_x \rangle$ may be calculated in multiple ways:

Reminder: Note that $|x(t)|^2 = x(t)\,x^*(t)$

(a) Energy within a time interval (t_1, t_2)

$$E_x = \int_{t_1}^{t_2} |x(t)|^2\ dt$$

(b) Total energy

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\ dt$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_3

(c) Average power within a time interval (t_1, t_2)

$$\langle P_x \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

(d) Average of total power

$$\langle P_x \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt$$

(e) Average power of a periodic signal within a period T_0

$$\langle P_x \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt$$

Problems

3.1 Basic Function Transformations

Given $x(t)$ function in Fig. 3.1, sketch graphs of functions as defined in P.3.1 to P.3.4.

3.1. $x(t-2)$ **3.2.** $x(-t)$ **3.3.** $x(2t)$ **3.4.** $x(0.2t)$

3.5. Given $x(t)$ function in Fig. 3.2, deduce and sketch graph of $y(t) = x(2t+3)$.

3.6. Given $x(t)$ function in Fig. 3.3, show graph of $y(t) = x(-t/2 - 4)$.

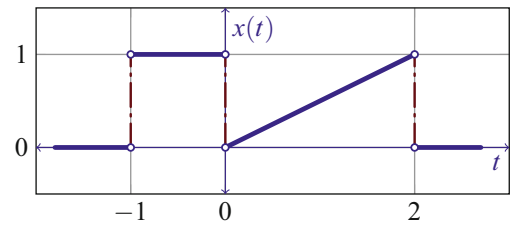


Fig. 3.1 Examples P.3.1 to P.3.4

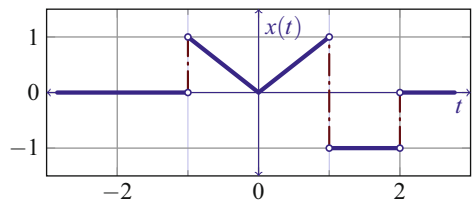


Fig. 3.2 Example P.3.5

Fig. 3.3 Example P.3.6

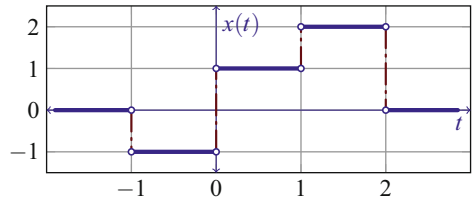
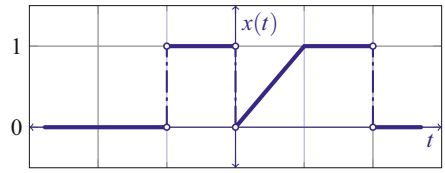


Fig. 3.4 Example P.3.7



3.2 Function Synthesis and Decomposition

Reminder: given arbitrary $x(t)$ function may be decomposed into sum of its even $x_p(t)$ and odd $x_i(t)$ functions that are defined as

$$x_p(t) = \frac{x(t) + x(-t)}{2} \tag{3.2}$$

$$x_i(t) = \frac{x(t) - x(-t)}{2} \tag{3.3}$$

Verification:

$$x_p(t) + x_i(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = \frac{x(t)}{2} + \frac{x(t)}{2} + \frac{x(-t)}{2} - \frac{x(-t)}{2}$$
$$= x(t)$$

Given functions in P.3.7 to P.3.11, do the required syntheses/decomposition operations.

3.7. Given $x(t)$ function in Fig. 3.4, decompose $x(t)$ into a sum of one even $x_p(t)$ and one odd $x_i(t)$ function so that $x(t) = x_p(t) + x_i(t)$, and then show their respective graphs.

3.8. Show the graphical representation of the following function:

$$x(t) = 3u(t) + r(t) - r(t-1) - 5u(t-2)$$

3.9. Derive the analytical form of function $x(t)$ given in Fig. 3.5.

Fig. 3.5 Example P.3.9

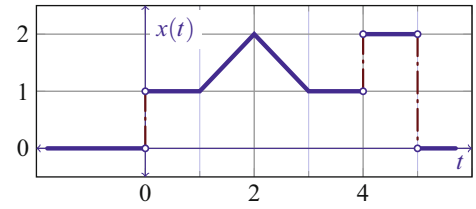


Fig. 3.6 Example P.3.10

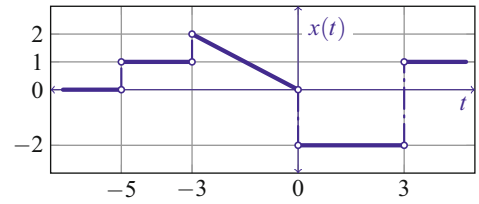
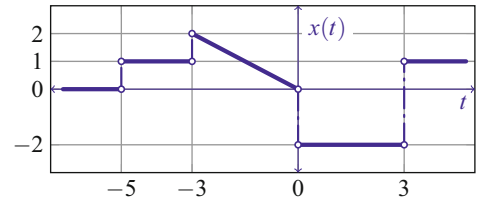


Fig. 3.7 Example P.3.11



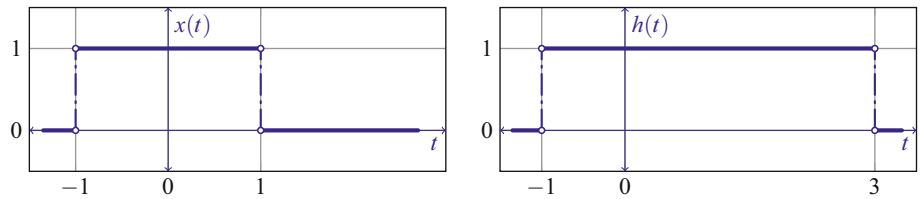


Fig. 3.8 Example P.3.12

3.10. Derive the analytical form of function $x(t)$ given in Fig. 3.6.

3.11. Derive the analytical form of function $x(t)$ given in Fig. 3.7.

3.3 Continuous Time Convolution

Resolve convolutions in P.3.12 to P.3.18.

3.12. Calculate $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are given in Fig. 3.8.

3.13. Calculate $z(t) = x(t) * y(t)$, where $x(t)$ and $y(t)$ are given in Fig. 3.9.

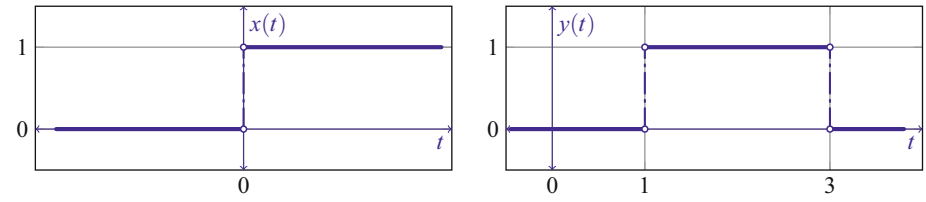


Fig. 3.9 Example P.3.13

3.14. Calculate $z(t) = u(t) * u(t)$.

3.15. Calculate $y(t) = x(t) * h(t)$, given

$$x(t) = \big(u(t+1) - u(t-1)\big)\, t$$
$$h(t) = \big(u(t+2) - u(t)\big)$$

3.16. Calculate $x(t) = \Lambda\,(t/2) * \Lambda\,(t/2)$.

3.17. Calculate $z(t) = x(t) * y(t)$, where $x(t) = t^3\, u(t)$ and $y(t) = t^2\, u(t)$.

3.18. Calculate $w(t) = x(t) * v(t)$, given

$$x(t) = \begin{cases} 1 - t; & 0 \le t \le 1 \\ 0; & \text{otherwise} \end{cases} \quad \text{and,} \quad v(t) = \begin{cases} e^{-t}; & t \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

3.4 Energy and Power

Given $x(t)$ functions, calculate their respective energies and powers in P.3.19 to P.3.33.

3.19. $x(t) = 2\Pi(t-1)$

3.20. $x(t) = \Lambda(t)$

3.21. $x(t) = \big(1/4 - t^2\big)\ \Pi(t)$

3.22. $x(t) = a \quad (a = \text{const.})$

3.23. $x(t) = u(t)$

3.24. $x(t) = u(-t)$

3.25. $x(t) = \sin(t)$

3.26. $x(t) = a\, e^{j\,\omega_0\, t}$

3.27. $x(t) = 1 + \cos(2t)$

3.28. $x(t) = e^{-t}\ u(t)$

3.29. $x(t) = e^{-t^2}$

3.30. $x(t) = e^{-|t|}$

3.31. $x(t) = r(t)$

3.32. $x(t) = e^{t}\, u(t)$

3.33. $x(t) = (1 - x^3)\, u(-t)$

Answers

3.1 Basic Function Transformations

3.1. After adding a constant $\pm a$ to its argument, the function does not change its original form while it translates along the horizontal axis by distance $\mp a$.

In the case of piecewise linear functions, it is sufficient to identify coordinates of some typical points, such as endpoints of each line segment, and to recalculate their new positions by replacing the original argument t with the new argument $t \pm a$. Given $x(t)$ in Fig. 3.1, one can notice changes (discontinuities) in the function's shape happening at $t = -1$, $t = 0$, and $t = 2$. Thus, to apply $x(t) \to x(t_o - 2)$ transformation, where t_o is the new t coordinate of each calculated point, is to say that by equalizing the original "(t)" and the new "$(t_o - 2)$" arguments, it follows that

t	$t = t_o - 2$	$\Rightarrow$	t_o
-1	$-1 = t_o - 2$		1
0	$0 = t_o - 2$		2
2	$2 = t_o - 2$		4

that is to say, all points are shifted in the positive direction by two units, see Fig. 3.10. Note the left and right side difference in limits at each point of discontinuity.

3.2. Following same reasoning as in P.3.1, given $x(t)$ in Fig. 3.1, it follows that after applying $x(t) \to x(-t_o)$ transformation the new coordinates of $x(t)$ are calculated as

t	$t = -t_o$	$\Rightarrow$	t_o
-1	$-1 = -t_o$		1
0	$0 = -t_o$		0
2	$2 = -t_o$		-2

that is to say, after multiplying function's argument by "-1" all points are mirrored ("time reversed") around the vertical axis, see Fig. 3.11.

3.3. Given $x(t)$ in Fig. 3.1 it follows that after applying $x(t) \to x(2\,t_o)$ transformation ("time compression") the new coordinates of $x(t)$ are calculated as

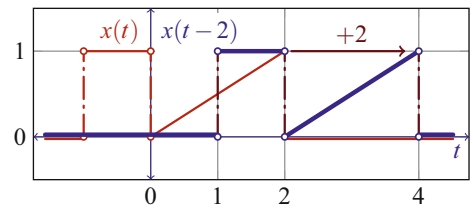


Fig. 3.10 Example P.3.1

Fig. 3.11 Example P.3.2

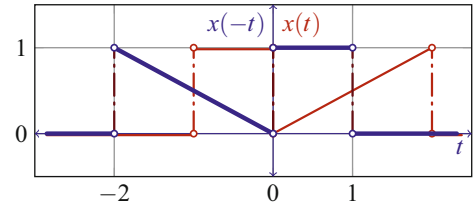


Fig. 3.12 Example P.3.3

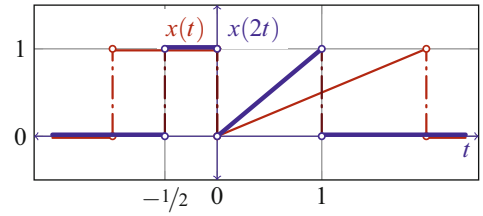
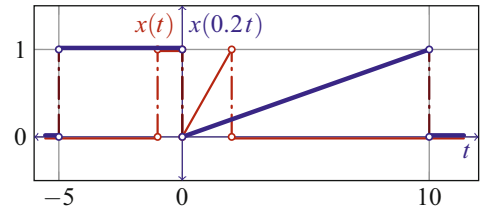


Fig. 3.13 Example P.3.4



t	$t = 2t_o$	$\Rightarrow$	t_o
-1	$-1 = 2t_o$		$-1/2$
0	$0 = 2t_o$		0
2	$2 = 2t_o$		1

that is to say, after multiplying function's argument by number greater than one, all coordinates are "time compressed" by the same factor, see Fig. 3.12. Note the left and right side limits at each point of discontinuity.

3.4. Given $x(t)$ in Fig. 3.1 it follows that after applying $x(t) \rightarrow x(0.2\,t_o)$ transformation ("time expansion") the new coordinates of $x(t)$ are calculated as

t	$t = 0.2t_o$	$\Rightarrow$	t_o
-1	$-1 = 0.2t_o$		-5
0	$0 = 0.2t_o$		0
2	$2 = 0.2t_o$		10

that is to say, after multiplying function's argument by number inferior to one (or, equivalently, dividing by a number superior to one), all coordinates are "time expanded" by the same factor, see Fig. 3.13.

3.5. Subsequent multiple transformations may be resolved by using various methods. Note the points of discontinuities at $t = -1, 0, 1, 2$ that are not difficult to follow visually. For example,

Method 1: Given $x(t)$ in Fig. 3.2 it follows that after applying $x(t) \to x(2\,t_o + 3)$ transformation all $x(t)$ points are moved to the new coordinates, that may be calculated directly as

t	$t = 2t_o + 3$	$\Rightarrow$	t_o
-1	$-1 = 2t_o + 3$		-2
0	$0 = 2t_o + 3$		$-3/2$
1	$1 = 2t_o + 3$		-1
2	$2 = 2t_o + 3$		$-1/2$

that is to say, all coordinates are "time compressed" and "shifted", see Fig. 3.14.

Method 2: Transformation $y(t) = x(2t + 3)$ consists of one compression (i.e $2t$) and one delay (i.e. $+3$). If the delay is applied before the compression, then

$$x(t) \xrightarrow{\text{delay}} a(t) = x(t + 3) \xrightarrow{\text{compression}} b(t) = a(2t)$$

In that case, the two step transformation is as follows

1. Applying delay $x(t + 3)$ results in (see Fig. 3.15)

t	$t = t_a + 3$	$\Rightarrow$	t_a
-1	$-1 = t_a + 3$		-4
0	$0 = t_a + 3$		-3
1	$1 = t_a + 3$		-2
2	$2 = t_a + 3$		-1

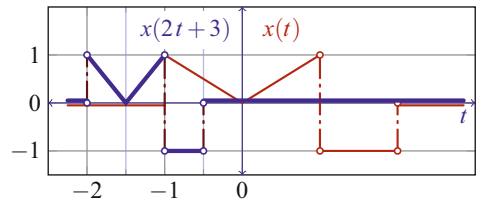


Fig. 3.14 Example P.3.5

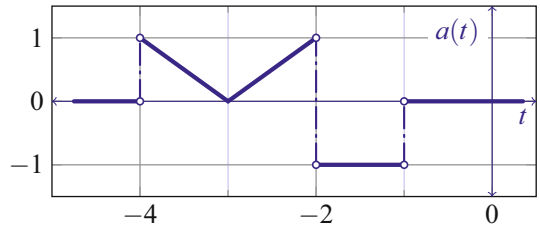


Fig. 3.15 Example P.3.5

2. Applying compression $a(2t_b)$ results in

t_a	$t_a = 2t_b$	$\Rightarrow$	t_b
-4	$-4 = 2t_b$		-2
-3	$-3 = 2t_b$		$-3/2$
-2	$-2 = 2t_b$		-1
-1	$-1 = 2t_b$		$-1/2$

which is same result $x(2t + 3)$ as already shown in Fig. 3.14.

Method 3: If compression is applied before delay, then the original delay must be compressed as well,

$$x(t) \xrightarrow{\text{compression}} a(t) = x(2t) \xrightarrow{\text{delay}} b(t) = a(t + 3) = x\left(2\left(t + \frac{3}{2}\right)\right)$$

In that case, the two step transformation is as follows,

1. Applying compression $x(2t)$ results in (see Fig. 3.16.)

t	$t = 2t_a$	$\Rightarrow$	t_a
-1	$-1 = 2t_a$		$-1/2$
0	$0 = 2t_a$		0
1	$1 = 2t_a$		$1/2$
2	$2 = 2t_a$		1

2. Applying delay $a(t + 3/2)$ results in

t_a	$t_a = t_b + 3/2$	$\Rightarrow$	t_b
$-1/2$	$-1/2 = t_b + 3/2$		-2
0	$0 = t_b + 3/2$		$-3/2$
$1/2$	$1/2 = t_b + 3/2$		-1
1	$1 = t_b + 3/2$		$-1/2$

which is in agreement with the previous two methods, as shown in Fig. 3.14.

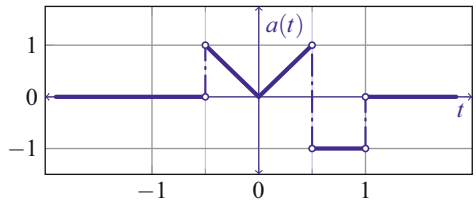
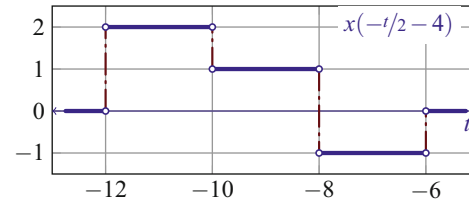


Fig. 3.16 Example P.3.5

Fig. 3.17 Example P.3.6



3.6. By following same reasoning as in A.3.5, subsequent multiple transformations may be resolved using various methods. For example,

Method 1: Given $x(t)$ in Fig. 3.3 note points of discontinuity at $t = -1, 0, 1, 2$, it follows that after applying $x(t) \to x(-t_o/2 - 4)$ transformation all $x(t)$ points are moved to the new coordinates, that may be calculated directly as

t	$t = -t_o/2 - 4$	$\Rightarrow$	t_o
-1	$-1 = -t_o/2 - 4$		-6
0	$0 = -t_o/2 - 4$		-8
1	$1 = -t_o/2 - 4$		-10
2	$2 = -t_o/2 - 4$		-12

that is to say, all coordinates are time "inverted", "expanded" and "shifted", see Fig. 3.17. As an exercise, this example may be solved using the other two methods in A.3.5.

3.2 Function Synthesis and Decomposition

3.7. Given function in Fig. 3.4 is piecewise linear, its even and odd functions (3.2) and (3.3) may be deduced by using graphical technique. That is to say, by adding or subtracting all line segments within each individual interval.

Two summing terms, $x(t)$ and $x(-t)$, that are needed to derive even function $x_p(t)$, are shown Fig. 3.18 (left). Summation of linear segments making these two terms is done interval by interval as follows,

t		-2		-1		0		1		2	
$x(t)$	0	$\underset{\to}{0}, \underset{\leftarrow}{0}$	0	$\underset{\to}{0}, \underset{\leftarrow}{1}$	1	$\underset{\to}{1}, \underset{\leftarrow}{0}$	t	$\underset{\to}{1}, \underset{\leftarrow}{1}$	1	$\underset{\to}{1}, \underset{\leftarrow}{0}$	0
$x(-t)$	0	$\underset{\to}{0}, \underset{\leftarrow}{1}$	1	$\underset{\to}{1}, \underset{\leftarrow}{1}$	$-t$	$\underset{\to}{0}, \underset{\leftarrow}{1}$	1	$\underset{\to}{1}, \underset{\leftarrow}{0}$	0	$\underset{\to}{0}, \underset{\leftarrow}{0}$	0
$x_p(t)$	0	$\underset{\to}{0}, \underset{\leftarrow}{1/2}$	$1/2$	$\underset{\to}{1/2}, \underset{\leftarrow}{1}$	$(1-t)/2$	$\underset{\to}{1/2}, \underset{\leftarrow}{1/2}$	$(1+t)/2$	$\underset{\to}{1}, \underset{\leftarrow}{1/2}$	$1/2$	$\underset{\to}{1/2}, \underset{\leftarrow}{0}$	0

For example, at $t = 0$ the left/right side summations $x(t) + x(-t)$ (see Fig. 3.18 (left)) are

$$\underset{\to}{t = 0}: \quad \frac{x(t) + x(-t)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

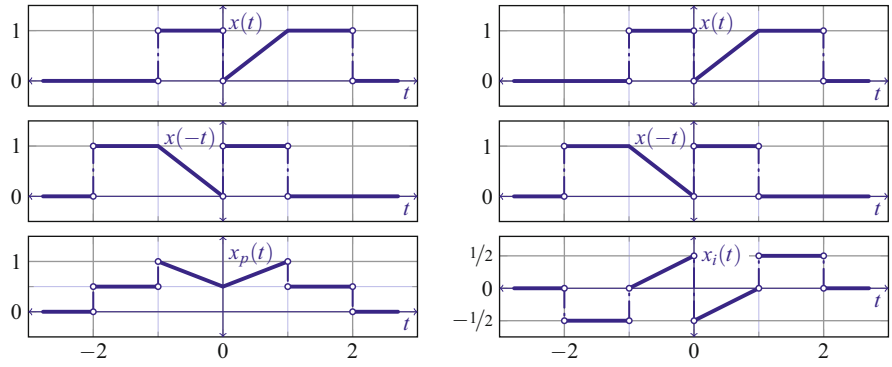


Fig. 3.18 Example P.3.7

$$\underset{\twoheleftarrow}{t=0}: \quad \frac{x(t)+x(-t)}{2}=\frac{0+1}{2}=\frac{1}{2}$$

The difference (3.3) between $x(t)$ and $x(-t)$ (see Fig. 3.18 (right)) is

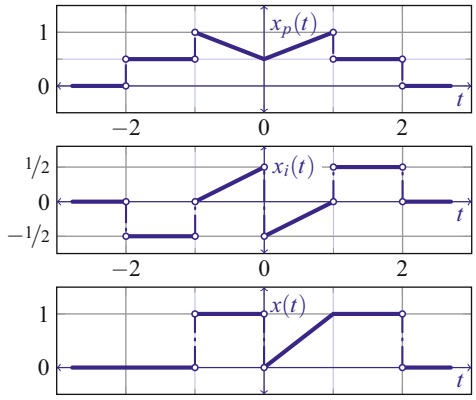
t		-2		-1		0		1		2	
$x(t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{0}$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{0}$	t	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{0}$	0
$x(-t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{1}$	$-t$	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{0}$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{0}$	0
$x_i(t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{-1/2}$	$-1/2$	$\underset{\twoheadrightarrow}{-1/2},\underset{\twoheleftarrow}{0}$	$(1+t)/2$	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{-1/2}$	$(t-1)/2$	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1/2}$	1/2	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{0}$	0

Verification: sum $x(t)=x_p(t)+x_i(t)$ is analytically confirmed as (see Fig. 3.19)

t		-2		-1		0		1		2	
$x_p(t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1/2}$	1/2	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{1}$	$(1-t)/2$	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{1/2}$	$(1+t)/2$	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{1/2}$	1/2	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{0}$	0
$x_i(t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{-1/2}$	$-1/2$	$\underset{\twoheadrightarrow}{-1/2},\underset{\twoheleftarrow}{0}$	$(1+t)/2$	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{-1/2}$	$(t-1)/2$	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1/2}$	1/2	$\underset{\twoheadrightarrow}{1/2},\underset{\twoheleftarrow}{0}$	0
$x(t)$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{0}$	0	$\underset{\twoheadrightarrow}{0},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{0}$	t	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{1}$	1	$\underset{\twoheadrightarrow}{1},\underset{\twoheleftarrow}{0}$	0

3.8. Function $x(t)$ may be synthesized by summation of the basic terms in the form of $u(t)$ and $r(t)$ functions. The individual terms needed to derive function $x(t)$ are shown Fig. 3.20 (left). Summation of all four terms is done interval by interval as follows,

Fig. 3.19 Example P.3.7



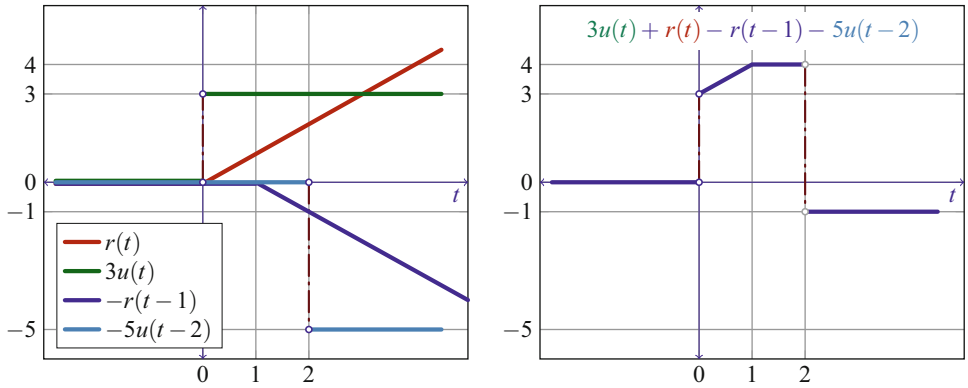


Fig. 3.20 Example P.3.8

t		0		1		2	
$3u(t)$	0	$0,\ 3$	3	$3,\ 3$	3	$3,\ 3$	3
$r(t)$	0	$0,\ 0$	t	$1,\ 1$	t	$2,\ 2$	t
$-r(t-1)$	0	$0,\ 0$	0	$0,\ 0$	$1-t$	$-1,\ -1$	$1-t$
$-5u(t-2)$	0	$0,\ 0$	0	$0,\ 0$	0	$0,\ -5$	-5
$x(t)$	0	$0,\ 3$	$3+t$	$4,\ 4$	4	$4,\ -1$	-1

and drafted in Fig. 3.20 (right). Note that the left and right side limits are not same at discontinuities.

3.9. There are infinite ways to synthesize function $x(t)$ given in Fig. 3.5, thus any possible analytical form that corresponds to $x(t)$ is not unique. Nevertheless, arguably simplest analytical form may be deduced by following changes interval by interval. For example:

1. Interval from minus infinity to minus one: $x(t)$ starts as $2u(t+2)$, see Fig. 3.21 (top). Then, at $t=-1$ its amplitude is reduced by one, which may be forced by adding $-u(t+1)$, see Fig. 3.21 (bottom).

Fig. 3.21 Example P.3.9

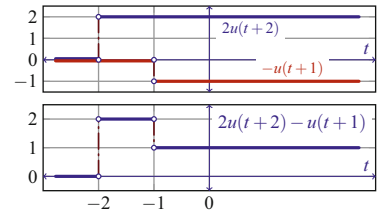


Fig. 3.22 Example P.3.9

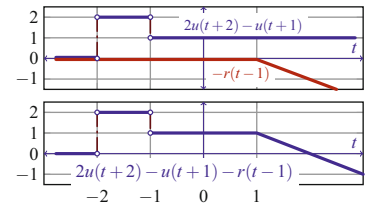


Fig. 3.23 Example P.3.9

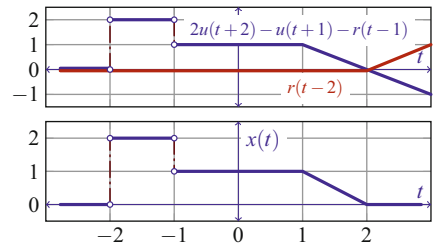
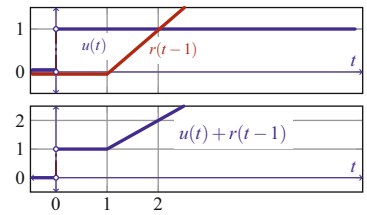


Fig. 3.24 Example P.3.10



2. The following change of $x(t)$ form happens at $t = 1$, which may be forced by adding ramp $-r(t-1)$, see Fig. 3.22 (top). If not cancelled, downward line segment of ramp of $-r(t-1)$ would continue to negative infinity, see Fig. 3.22 (bottom).
3. In order to cancel $-r(t-1)$ ramp after $t = 2$ point, it is necessary to add positive ramp with the same slope, i.e

$$-r(t-1) + r(t-2) = 0, \quad (x \geq 2)$$

as is evident in Fig. 3.23 (bottom) so that compete synthesis of $x(t)$ is

$$x(t) = 2u(t+2) - u(t+1) - r(t-1) + r(t-2)$$

3.10. There are infinite ways to synthesize function $x(t)$ given in Fig. 3.6, for example:

Fig. 3.25 Example P.3.10

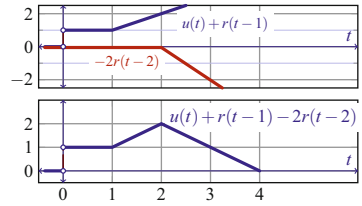


Fig. 3.26 Example P.3.10

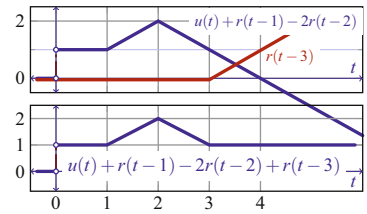
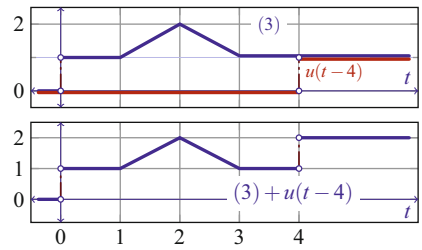


Fig. 3.27 Example P.3.10



1. Interval from minus infinity to minus one: $x(t)$ starts as $u(t)$ function followed by delayed ramp $r(t-1)$, see Fig. 3.24 (top). If not cancelled, upward line segment of ramp of $r(t-1)$ would continue to negative infinity, see Fig. 3.24 (bottom).
2. The following change of $x(t)$ form is at $t=2$, which may be forced by adding delayed ramp whose slope is *two times negative*, i.e. $-2r(t-2)$, see Fig. 3.25 (top). Note that it is necessary to add two ramp functions $-r(t-2)$ so that ramp $r(t-1)$ for $x \le 2$ is replaced by $r(t-1) - 2r(t-2) = -r(t-2), \quad t \ge 2$, see Fig. 3.25 (bottom).
3. If not cancelled, downward line segment of ramp of $-r(t-2)$ would continue to negative infinity. In order to cancel it after $t=3$ point, it is necessary to add positive ramp with the same slope (see Fig. 3.26 (top)), i.e

$$-r(t-2) + r(t-3) = 0, \quad (x \ge 3)$$

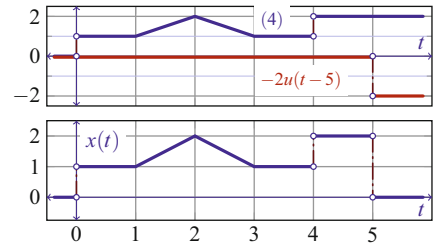
   as is evident in Fig. 3.26 (bottom), thereafter $x(t) = 1$ is constant.
4. As the next summation term, delayed positive step function $u(t-4)$ is added after $t=4$ point to the form derived in (3), as is evident in Fig. 3.27.
5. Finally, form derived in (4) is added to negative step $-u(t-5)$ at $t=5$ to complete $x(t)$, see Fig. 3.28.

In conclusion, one possible analytical form is

$$x(t) = u(t) + r(t-1) - 2r(t-2) + r(t-3) + u(t-4) - 2u(t-5)$$

Note how $x(t)$ is formed in $t \in [1,3]$ interval by adding "$r(t-1) - 2r(t-2) + r(t-3)$" ramp functions.

Fig. 3.28 Example P.3.10



3.11. Following the same reasoning as in P.3.10, by inspection of graph in Fig. 3.7, it may be deduced that,

$$f(t) = u(t+5) + u(t+3) - \frac{2}{3}r(t+3) + \frac{2}{3}r(t) - 2u(t) + 3u(t-3)$$

It may be verified, for example at $x = \underset{\rightarrow}{0}$, i.e. from the left side, as

$$\begin{aligned} f(\underset{\rightarrow}{0}) &= u(\underset{\rightarrow}{0}+5) + u(\underset{\rightarrow}{0}+3) - \frac{2}{3}r(\underset{\rightarrow}{0}+3) + \frac{2}{3}r(\underset{\rightarrow}{0}) - 2u(\underset{\rightarrow}{0}) + 3u(\underset{\rightarrow}{0}-3) \\ &= 1 + 1 - \frac{2}{\cancel{3}}(\cancel{3}) + \frac{2}{3}(0) - \underline{2\,(0)} + 3\,(0) = \underset{\rightarrow}{0} \end{aligned}$$

which is correct coordinate of the point $(\underset{\rightarrow}{0}, 0)$ in Fig. 3.7. Similarly, from the right side of $t = 0$ it is calculated that

$$\begin{aligned} f(\underset{\leftarrow}{0}) &= u(\underset{\leftarrow}{0}+5) + u(\underset{\leftarrow}{0}+3) - \frac{2}{3}r(\underset{\leftarrow}{0}+3) + \frac{2}{3}r(\underset{\leftarrow}{0}) - 2u(\underset{\leftarrow}{0}) + 3u(\underset{\leftarrow}{0}-3) \\ &= 1 + 1 - \frac{2}{\cancel{3}}(\cancel{3}) + \frac{2}{3}(0) - \underline{2\,(1)} + 3\,(0) = \underset{\leftarrow}{-2} \end{aligned}$$

Note left and right side values of $-2u(0)$ term, as underlined above.

3.3 Continuous Time Convolution

Convolution integral (3.1) consists of the $x(\tau)h(t-\tau)$ product. Therefore, in the preparation part, it is necessary to create these two functions by the change of variables method and then to resolve their product. First, the change of variable $x(t) \rightarrow x(\tau)$ does not change the $x(t)$ function's form. Then, it is necessary to perform a sequence of $h(t)$ transformations, that is to say, to convert $h(t) \rightarrow h(\tau) \rightarrow h(-\tau) \rightarrow h(t-\tau)$ function. As the change of variable t into provisional integration variable τ does not change the original forms of neither $x(t)$ nor $h(t)$ functions, only transformations of $h(t) \rightarrow h(t-\tau)$ function are illustrated.

3.12. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Fig. 3.29 Example P.3.12

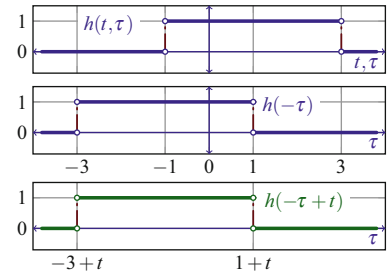
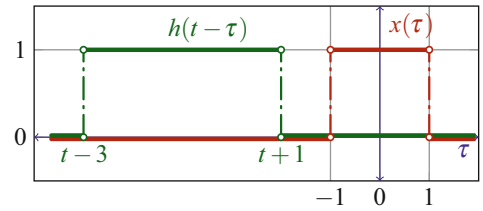


Fig. 3.30 Example P.3.12



Preparatory work : $h(t) \to h(\tau) \to h(-\tau) \to h(t-\tau)$ transformations.

(a) $h(t) \to h(\tau)$ transformation does not change form, only the variable $t \to \tau$, Fig. 3.29 (top).
(b) $h(\tau) \to h(-\tau)$ time inversion transformation results in the mirrored form, Fig. 3.29 (middle). Note that $\tau = 0$ reference point is still known.
(c) $h(-\tau) \to h(t-\tau)$ transformation shifts each point of $h(-\tau)$ by t, Fig. 3.29 (bottom). Note that $\tau = 0$ reference point is not known, it is relative to t, thus is resolved case by case.

Case 1: $(t+1) \leq -1$:

The leading edge of $h(t-\tau)$ is inferior to $(\tau = -1)$, in other words, there is no overlapping area between $h(t-\tau)$ and $x(\tau)$

$$t + 1 \leq -1 \quad \therefore \quad t \leq -2$$

That being the case, it is irrelevant where exactly $(t+1)$ point is relative to the trailing edge of $x(\tau)$ at $(\tau = -1)$, see Fig. 3.30. What is important is that under $(t \leq -2)$ condition, due to either $h(t-\tau)$ or $x(\tau)$ equal zero, *their product always equal zero*

$$x(\tau)\ h(t-\tau) = 0, \quad \forall \tau$$

and by consequence, the convolution integral also equal zero,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau = 0 \quad (t \leq -2)$$

Case 2: $-1 < (t+1) \leq 1$:

The leading edge of $h(t-\tau)$ is found within $\tau \in (-1, 1)$ interval bound by $x(\tau)$, however the trailing edge of $h(t-\tau)$ is still outside of this interval (note that $h(t-\tau)$ rectangle is longer than

$x(\tau)\big)$, i.e.

$$\left.\begin{array}{ll} t+1>-1 & \therefore\ t>-2 \\ t+1\le 1 & \therefore\ t\le 0 \end{array}\right\} \ \Rightarrow\ \underline{-2<t\le 0}$$

Therefore, within interval $\tau\in(-1,t+1)$, see Fig. 3.31, there is the overlapping non–zero area because that is where $h(t-\tau)=1$ and $x(\tau)=1$, therefore the convolution integral is

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau=\int_{-1}^{t+1}(1)\ (1)\ d\tau=\tau\Big|_{-1}^{t+1}=t+1-(-1)$$
$$=t+2\ \ (-2\le t\le 0)$$

Case 3 : $(t+1)>1$ and $(t-3)\le -1$:

The leading edge of $h(t-\tau)$ is found at $(t+1>1)$ while its trailing edge is still $(t-3\le -1)$, i.e.

$$\left.\begin{array}{ll} t+1>1 & \therefore\ t>0 \\ t-3\le -1 & \therefore\ t\le 2 \end{array}\right\} \ \Rightarrow\ \underline{0<t\le 2}$$

Note that $h(t-\tau)$ rectangle is longer than $x(\tau)$. Under this condition, see Fig. 3.32, the overlapping area is found in the interval $\tau\in(-1,+1)$ where $h(t-\tau)=1$ and $x(\tau)=1$, in other words it is maximal possible overlapping rectangular surface whose area equals two,

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau=\int_{-1}^{1}(1)\ (1)\ d\tau=\tau\Big|_{-1}^{1}$$
$$=2\ \ (0\le t\le 2)$$

Case 4 : $(t-3)\ge -1$ and $(t-3)\le 1$:

The trailing edge at $(t-3)$ of $h(t-\tau)$ entered the $\tau\in(-1,1)$ interval bound by $x(\tau)$, which is to say that the leading edge at $(t+1)$ of $h(t-\tau)$ must be somewhere at $t>1$, see Fig. 3.33, i.e.

$$\left.\begin{array}{ll} t-3\ge -1 & \therefore\ t\le 2 \\ t-3<1 & \therefore\ t<4 \end{array}\right\} \ \Rightarrow\ \underline{2\le t<4}$$

therefore, the overlapping area is found in the interval $\tau\in(t-3,+1)$ where both $h(t-\tau)=1$ and $x(\tau)=1$, i.e.

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau=\int_{t-3}^{1}(1)\ (1)\ d\tau=\tau\Big|_{t-3}^{1}$$
$$=4-t\ \ (2\le t<4)$$

Fig. 3.31 Example P.3.12

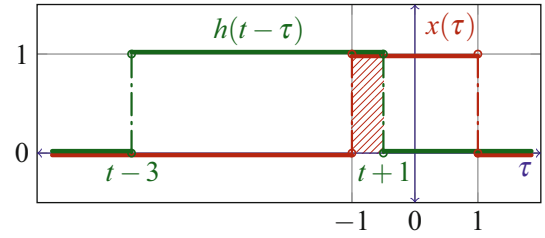


Fig. 3.32 Example P.3.12

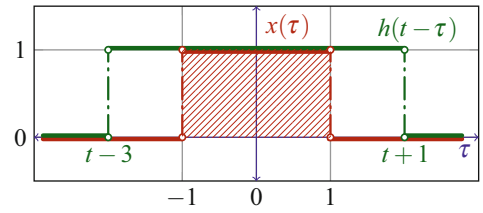
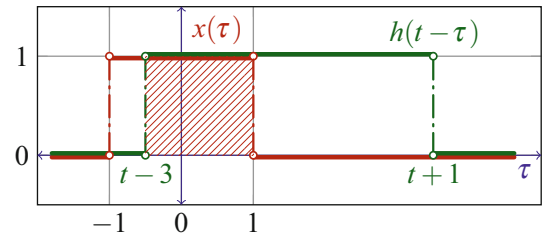


Fig. 3.33 Example P.3.12



Case 5 : $(t-3) \geq 1$:

The trailing edge at $(t-3)$ of $h(t-\tau)$ passed completely the $x(\tau)$ region, i.e. $(t-3 \geq 1)$, see Fig. 3.34, i.e.

$$t-3 \geq 1 \quad \therefore \quad t \geq 4$$

That being the case, it is irrelevant where exactly $(t-3)$ is relative to the leading edge of $x(\tau)$ at $\tau = 1$, see Fig. 3.34. What is important is that under this condition, due to either $h(t-\tau)$ or $x(\tau)$ equals zero, their product is therefore zero

$$x(\tau)\, h(t-\tau) = 0, \quad \forall \tau \qquad \therefore$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\; h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau = 0 \quad (t \geq 4)$$

Fig. 3.34 Example P.3.12

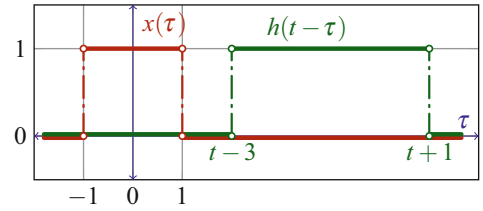
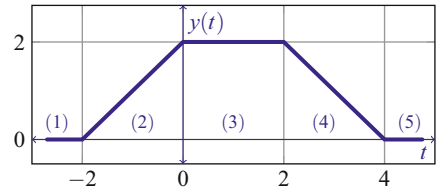


Fig. 3.35 Example P.3.12



Summary: The total convolution integral $y(t)$ shows the progression of the overlapping surface area between $x(\tau)$ and $h(t-\tau)$ in function of t, see Fig. 3.35, where within each interval the solution function is different, as resolved in cases (1) to (5). Note that the use of '$\leq$' and '$<$' symbols to bound the intervals is not strict because $y(t)$ is continuous, thus the interval bound points may be assigned either on the left or right side of the inequalities, as

$$y(t) = \begin{cases} 0 & t \leq -2 \\ 2+t & -2 \leq t \leq 0 \\ 2 & 0 \leq t \leq 2 \\ 4-t & 2 \leq t \leq 4 \\ 0 & t \geq 4 \end{cases}$$

3.13. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Preparatory work : $y(t) \to y(\tau) \to y(-\tau) \to y(t-\tau)$ transformations.

(a) $y(t) \to y(\tau)$ transformation does not change form, only the variable $t \to \tau$, Fig. 3.36 (top).
(b) $y(\tau) \to y(-\tau)$ time inversion transformation results in the mirrored form, Fig. 3.36 (middle). Note that $\tau = 0$ reference point is still known.
(c) $y(-\tau) \to y(t-\tau)$ transformation shifts each point of $y(-\tau)$ by t, Fig. 3.36 (bottom). Note that $\tau = 0$ reference point is not known, it is relative to t, thus is resolved case by case.

Case 1 : $(t-1) \leq 0$:

The leading edge $(t-1)$ of $y(t-\tau)$ is inferior to $\tau = 0$ i.e.

$$t - 1 \leq 0 \quad \therefore \quad t \leq 1$$

Fig. 3.36 Example P.3.13

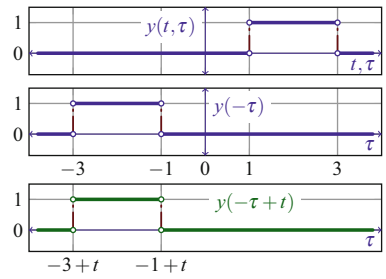
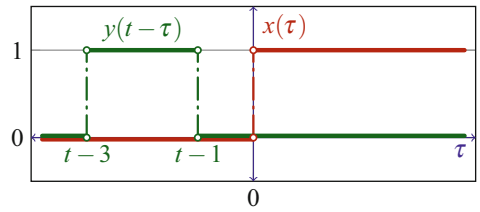


Fig. 3.37 Example P.3.13



That being the case, it is irrelevant where exactly t is relative to the origin, see Fig. 3.37. What is important is that under this condition, because always either of the two function equals zero, their product equal zero, as

$$x(\tau)\ y(t-\tau) = 0, \quad \forall\tau$$

and by consequence the convolution integral equal zero, as

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau \\ &= 0 \quad (t \le 1) \end{aligned}$$

Case 2 : $(t-1) > 0$ and $(t-3) \le 0$:

The leading edge at $(t-1)$ of $y(t-\tau)$ passed into the positive side, however the trailing edge at $(t-3)$ of $y(t-\tau)$ is still in the negative side, i.e.

$$\left.\begin{aligned} t-1 > 0 \quad &\therefore \quad t > 1 \\ t-3 \le 0 \quad &\therefore \quad t \le 3 \end{aligned}\right\} \quad \Rightarrow \quad \underline{1 < t \le 3}$$

Under this condition, the overlapping area is found in the interval $\tau \in (0, t-1)$, see Fig. 3.38, where both $y(t-\tau) = 1$ and $x(\tau) = 1$, therefore the convolution integral (i.e. the overlapping surface area) equal

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\, d\tau = \int_{0}^{t-1} (1)\ (1)\, d\tau = \tau \Big|_{0}^{t-1} \\ &= t-1 \quad (1 \le t \le 3) \end{aligned}$$

Fig. 3.38 Example P.3.13

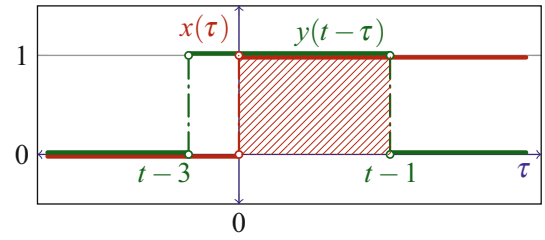


Fig. 3.39 Example P.3.13

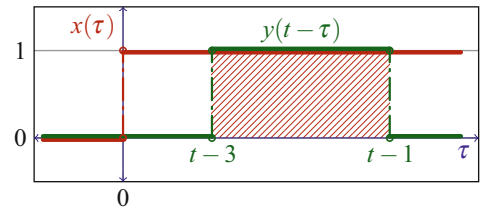
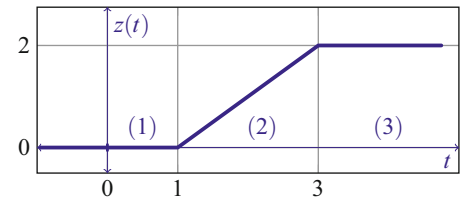


Fig. 3.40 Example P.3.13



Case 3 : $(t-3) > 0$:

The trailing edge $(t-3)$ of $y(t-\tau)$ shifted into the positive side, see Fig. 3.39, i.e.

$$t - 3 > 0 \quad \therefore \quad t > 3$$

Therefore, the overlapping surface area is at its maximum (i.e. the full area of rectangular function, and henceforth stays constant)

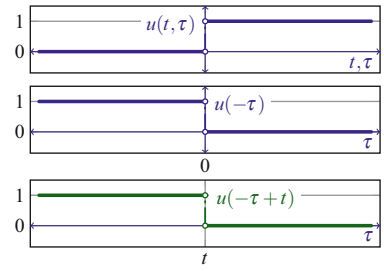
$$z(t) = \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\, d\tau = \int_{t-3}^{t-1} (1)\ (1)\, d\tau = \tau \Big|_{t-3}^{t-1} = (t-1) - (t-3)$$
$$= 2 \quad (t > 3)$$

Summary:

The total convolution integral shows progression of the overlapping area in function of t as resolved in the cases (1) to (3), Fig. 3.40, as

$$z(t) = \begin{cases} 0 & t \le 1 \\ t-1 & 1 \le t \le 3 \\ 2 & t > 3 \end{cases}$$

Fig. 3.41 Example P.3.14



3.14. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Preparatory work : $u(t) \to u(\tau) \to u(-\tau) \to u(t-\tau)$ transformations.

(a) $u(t) \to u(\tau)$ transformation does not change form, only the variable $t \to \tau$, Fig. 3.41 (top).
(b) $u(\tau) \to u(-\tau)$ time inversion transformation results in the mirrored form, Fig. 3.41 (middle). Note that $\tau = 0$ reference point is still known.
(c) $u(-\tau) \to u(t-\tau)$ transformation shifts each point of $u(-\tau)$ by t, Fig. 3.41 (bottom). Note that $\tau = 0$ reference point is not known, it is relative to t, thus is resolved case by case.

There are two distinct cases to solve.

Case 1 : $t \leq 0$:

The leading edge of $u(t-\tau)$ is still in the negative side, i.e.

$$t \leq 0$$

It is irrelevant where exactly t is relative to the origin, see Fig. 3.42. What is important is that under this condition, because always either of the two function equals zero, their product equal zero as

$$u(\tau)\, u(t-\tau) = 0, \quad \forall \tau$$

by consequence, convolution integral equal zero as,

$$z(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau$$
$$= 0 \quad (t \leq 0)$$

Case 2 : $t > 0$:

The leading edge of $u(t-\tau)$ is shifted into the positive side, Fig. 3.43, i.e.

$$t > 0$$

Fig. 3.42 Example P.3.14

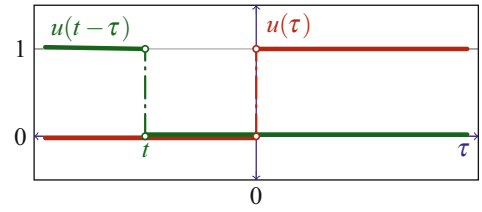


Fig. 3.43 Example P.3.14

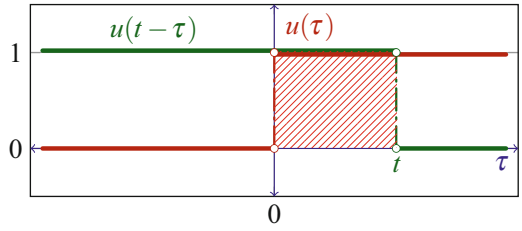
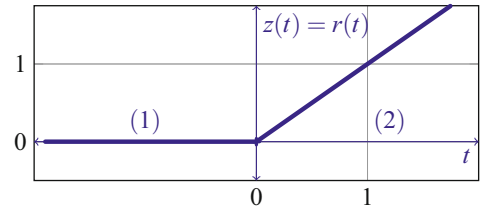


Fig. 3.44 Example P.3.14



thus,

$$z(t) = \int_{-\infty}^{\infty} u(\tau)\; u(t-\tau)\; d\tau = \int_{0}^{t} (1)\; (1)\; d\tau = \tau \Big|_{0}^{t}$$
$$= t \quad (t > 0)$$

(Rectangular area is calculated as height (equal one) times its length (equal t).)
Summary:

The total convolution integral is therefore $r(t)$ function, see Fig. 3.44, or written in equally elegant product form of linear and step functions as

$$z(t) = \begin{cases} 0 & t \le 0 \\ t & t > 0 \end{cases} \quad = \quad r(t) \quad \text{or,} \quad z(t) = t\, u(t)$$

3.15. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

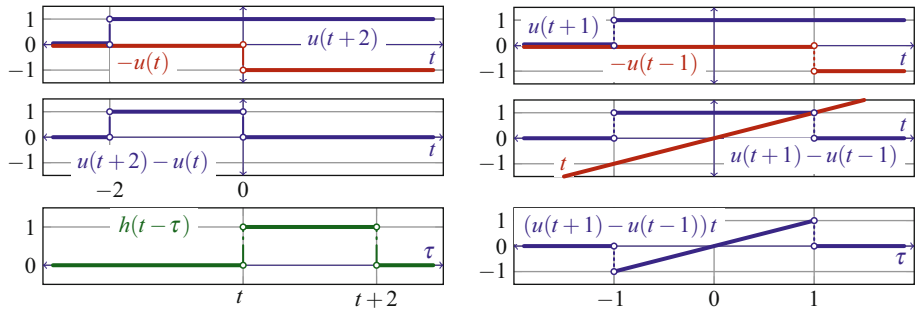
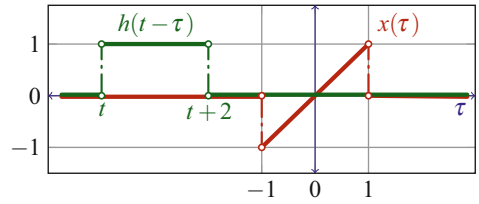


Fig. 3.45 Example P.3.15

Fig. 3.46 Example P.3.15



Preparatory work :

(a) the syntheses of $h(t) = \big(u(t+2) - u(t)\big)$ and $h(t-\tau)$ is illustrated in Fig. 3.45 (left).
(b) the syntheses of $x(t) = \big(u(t+1) - u(t-1)\big)\, t$ is illustrated in Fig. 3.45 (right)

There are four distinct cases to solve.

Case 1 : $t + 2 \leq -1$:

The leading edge of $h(t-\tau)$ is inferior to $\tau = -1$, that is to say

$$t + 2 \leq -1 \quad \therefore \quad t \leq -3$$

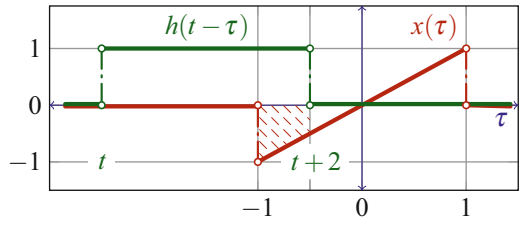
That being the case, it is irrelevant where exactly t is relative to $\tau = -1$, see Fig. 3.46. What is important is that under this condition, because either of the two function equals zero, their product equal zero as

$$x(\tau)\; h(t-\tau) = 0, \quad \forall \tau$$

and by consequence the convolution integral equal zero as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\; h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau$$
$$= 0 \quad (t \leq -3)$$

Fig. 3.47 Example P.3.15



Case 2 : $t+2 > -1$ and $t+2 \leq 1$

The leading edge of $h(t-\tau)$ is found within the interval $\tau \in (-1, 1)$, that is to say

$$\left.\begin{array}{ll} t+2 > -1 & \therefore \; t > -3 \\ t+2 \leq 1 & \therefore \; t \leq -1 \end{array}\right\} \;\Rightarrow\; \underline{-3 < t \leq -1}$$

Under this condition, see Fig. 3.47, the overlapping surface is found in the interval $\tau \in (-1, t+2)$ where $h(t-\tau) = 1$ and $x(\tau) = t$, consequently

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\; h(t-\tau)\; d\tau = \int_{-1}^{t+2} (\tau)\; (1)\; d\tau = \frac{1}{2}\; \tau^2 \Big|_{-1}^{t+2} \\ &= \frac{1}{2}\left[(t+2)^2 - (-1)^2\right] = \frac{1}{2}\left(t^2 + 4t + 3\right) \\ &= \frac{1}{2}(t+1)(t+3), \quad (-3 < t \leq -1) \end{aligned}$$

Note that both $h(t-\tau)$ and $x(\tau)$ non–zero intervals are equal, both lengths equal two.

Case 3 : $t > -1$ and $t \leq 1$

The trailing edge of $h(t-\tau)$ is found within the interval $\tau \in (-1, 1)$, that is to say

$$\underline{-1 < t \leq 1}$$

Under this condition, see Fig. 3.48, the overlapping surface is found in the interval $\tau \in (t, 1)$ where $h(t-\tau) = 1$ and $x(\tau) = t$, consequently

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\; h(t-\tau)\; d\tau = \int_{t}^{1} (\tau)\; (1)\; d\tau = \frac{1}{2}\; \tau^2 \Big|_{t}^{1} \\ &= \frac{1}{2}(1 - t^2) = \left\{(a^2 - b^2) = (a-b)(a+b)\right\} \\ &= -\frac{1}{2}(t-1)(t+1), \quad (-1 < t \leq 1) \end{aligned}$$

Fig. 3.48 Example P.3.15

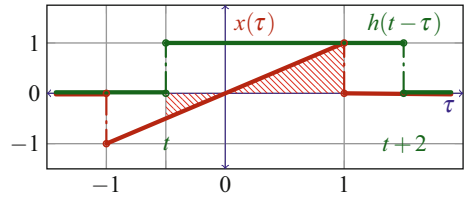
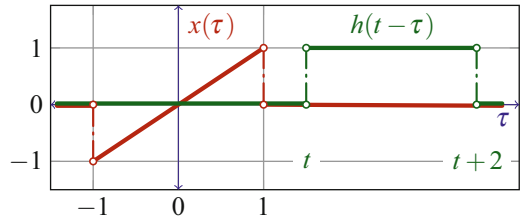


Fig. 3.49 Example P.3.15



Case 4 : $t > 1$

The trailing edge of $h(t-\tau)$ is superior to $\tau = 1$, that is to say

$$t > 1$$

That being the case, it is irrelevant where exactly t is relative to $\tau = 1$, see Fig. 3.49. What is important is that under this condition either of the two function equal zero, therefore their product equal zero as,

$$x(\tau)\ h(t-\tau) = 0, \quad \forall\tau$$

and by consequence the convolution integral equal to zero as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau \\ &= 0 \quad (t > 1) \end{aligned}$$

Summary:

The total convolution integral is therefore $y(t)$ function, see Fig. 3.50, as

$$y(t) = \begin{cases} 0 & t < -3 \\ \frac{1}{2}(t+1)(t+3) & (-3 \le t \le -1) \\ -\frac{1}{2}(t-1)(t+1) & (-1 \le t \le 1) \\ 0 & t > 1 \end{cases}$$

Fig. 3.50 Example P.3.15

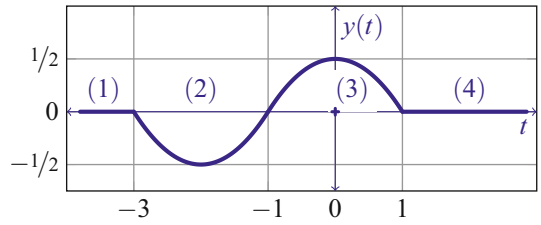
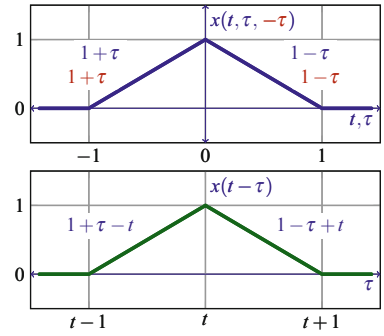


Fig. 3.51 Example P.3.16



3.16. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Preparatory work : $x(t) \to x(\tau) \to x(-\tau) \to x(t-\tau)$ transformations.

(a) $x(t) \to x(\tau)$ transformation does not change form, only the variable $t \to \tau$. However, triangular function is even and by consequence $x(\tau) \to x(-\tau)$ time inversion transformation also does not change its form.
Even after $x(-\tau) \to x(t-\tau)$ transformation, visually, triangular form does not change, see Fig. 3.51.

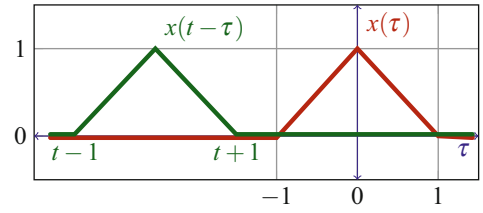
Although changes are not evident visually, note how time inversion alters linear equations on $(\tau \geq 0)$ and $(\tau \leq 0)$ sides

$$x(\tau) = 1+\tau \quad \Rightarrow \quad x(-\tau) = 1-\tau$$
$$x(\tau) = 1-\tau \quad \Rightarrow \quad x(-\tau) = 1-(-\tau) = 1+\tau$$

(b) It is important, however, to explicitly write linear equations that correspond to each linear segment after $x(-\tau) \to x(t-\tau)$ transformation as:

$$\left.\begin{array}{lll} x(-\tau) = & 1+\tau, & -1 \geq \tau \geq 0 \\ x(-\tau) = & 1-\tau, & 0 \leq \tau \leq 1 \\ x(-\tau) = & 0, & |\tau| \geq 1 \end{array}\right\} \quad \therefore \quad \left\{\begin{array}{lll} x(t-\tau) = & 1+\tau-t, & t-1 \leq \tau \leq t \\ x(t-\tau) = & 1-\tau+t, & t \leq \tau \leq t+1 \\ x(t-\tau) = & 0, & \text{otherwise} \end{array}\right.$$

Fig. 3.52 Example P.3.16



Reminder: Note that for $\tau \geq 0$ shifting by '$+t$' means adding '$+t$' so that $(1-\tau) \rightarrow (1-\tau+t)$, while for $\tau \leq 0$ shifting by '$+t$' means adding '$-t$' so that $(1+\tau) \rightarrow (1+\tau-t)$.

There are six distinct cases to solve.

Case 1 : $t+1 < -1$

The leading $(t+1)$ point of $x(t-\tau)$ is inferior to $\tau = -1$, i.e.

$$t+1 < -1 \quad \therefore \quad \underline{t < -2}$$

That being the case, it is irrelevant where exactly t is relative to the origin, see Fig. 3.52. What is important is that under this condition at least one of the two functions equal zero, therefore their product equal zero as

$$x(\tau)\ x(t-\tau) = 0, \quad \forall\tau$$

and by consequence, convolution integral equal zero,

$$z(t) = \int_{-\infty}^{\infty} x(\tau)\ x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 0\, d\tau$$
$$= 0 \quad (t < -2)$$

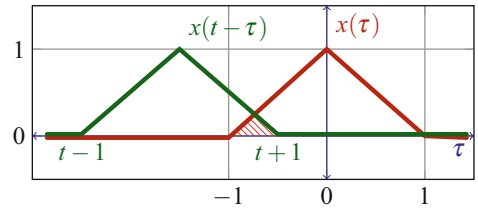
Case 2 : $t+1 \geq -1, \quad t+1 \leq 0$

As long as the leading $(t+1)$ point of $x(t-\tau)$ is in $\tau \in (-1, 0)$ interval, i.e.

$$\left.\begin{array}{ll} t+1 \geq -1 & \therefore\ t \geq -2 \quad \text{and} \\ t+1 \leq 0 & \therefore\ t < -1 \end{array}\right\} \ \therefore \quad \underline{-2 \leq t < -1}$$

the overlapping surface is bound by $(1-\tau+t)$ and $(1+\tau)$ linear segments, and $(-1, t+1)$ interval, see Fig. 3.53. Thus, product of these two linear segments defines convolution integral as:

Fig. 3.53 Example P.3.16



$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau = \int_{-1}^{t+1} (1+\tau)\,(1-\tau+t)\, d\tau \\
&= \int_{-1}^{t+1} (1 - \cancel{\tau} + t + \cancel{\tau} + t\tau - \tau^2)\, d\tau \\
&= (1+t)\,\tau \Big|_{-1}^{t+1} + \frac{t}{2}\,\tau^2 \Big|_{-1}^{t+1} - \frac{1}{3}\,\tau^3 \Big|_{-1}^{t+1} \\
&= (1+t)\big((t+1)-(-1)\big) + \frac{t}{2}\big((t+1)^2-(-1)^2\big) - \frac{1}{3}\big((t+1)^3-(-1)^3\big) \\
&= \{\text{use Pascal triangle for binomial power development}\} \\
&= \frac{(t+2)^3}{6}
\end{aligned}
$$

after factorizing third order polynomial.

Case 3 : $t+1 \geq 0, \quad t+1 \leq 1$

The leading $(t+1)$ point of $x(t-\tau)$ is found in $\tau \in (0,1)$ interval, i.e.

$$
\left.
\begin{array}{lll}
t+1 > 0 & \therefore & t > -1 \quad \text{and} \\
t+1 \leq 1 & \therefore & t < 0
\end{array}
\right\} \quad \therefore \quad -1 < t \leq 0
$$

where, overlapping surface is found in $\tau \in (-1, t+1)$ interval, Fig. 3.54. However, this time, actually there are three distinct sub–intervals (a), (b), (c), as

$$
\begin{aligned}
(a)\ &:(-1, t), \quad (1+\tau)\,(1+\tau-t) \\
(b)\ &:(t, 0), \quad (1+\tau)\,(1-\tau+t) \\
(c)\ &:(0, t+1), \quad (1-\tau)\,(1-\tau+t)
\end{aligned}
$$

where the convolution products found in each of sub–intervals are different, see Fig. 3.54, and the convolution integral is split into the sum of three integrals so that the overlapping areas are correctly calculated as

Fig. 3.54 Example P.3.16

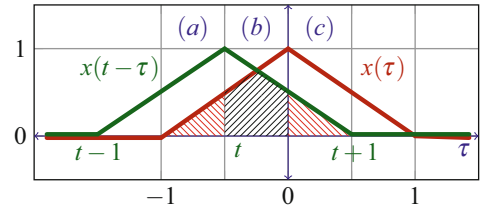
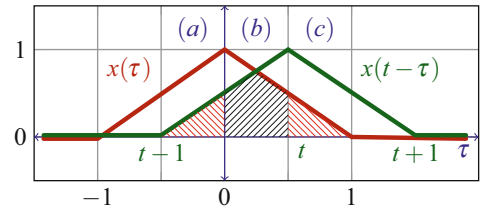


Fig. 3.55 Example P.3.16



$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau = \int_{-1}^{t} (1+\tau)\, (1+\tau-t)\, d\tau \\
&\quad + \int_{t}^{0} (1+\tau)\, (1-\tau+t)\, d\tau \\
&\quad + \int_{0}^{t+1} (1-\tau)\, (1-\tau+t)\, d\tau \\
&= \cdots = \frac{1}{6}\,(-t^3+3t+2) \\
&\quad - \frac{1}{6}\, t\,(t^2+6t+6) \\
&\quad + \frac{1}{6}\,(-t^3+3t+2) \\
&= \frac{1}{3}(-t^3+3t+2) - \frac{1}{6}\, t\,(t^2+6t+6), \quad (-1 < t \le 0)
\end{aligned}
$$

Case 4 : $t-1 \ge -1, \quad t-1 \le 0$

As long as the trailing $(t-1)$ point of $x(t-\tau)$ is in $\tau \in (-1, 0)$ interval, i.e.

$$
\left.
\begin{aligned}
t-1 \ge -1 \quad &\therefore \quad t > 0 \\
t-1 < 0 \quad &\therefore \quad t < 1
\end{aligned}
\right\} \quad \therefore \quad 0 < t \le 1
$$

where, overlapping surface is found in $\tau \in (t-1, 1)$ interval, Fig. 3.55. There are three distinct sub–intervals (a), (b), (c),

$$(a) : (t-1, 0), \quad (1+\tau) \quad (1+\tau-t)$$
$$(b) : (0, t), \quad (1-\tau) \quad (1+\tau-t)$$
$$(c) : (t, 1), \quad (1-\tau) \quad (1-\tau+t)$$

and the convolution integral is split into the sum of three integrals so that the overlapping areas are correctly calculated as

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau = \int_{t-1}^{0} (1+\tau) \quad (1+\tau-t)\, d\tau \\ &+ \int_{0}^{t} (1-\tau) \quad (1+\tau-t)\, d\tau \\ &+ \int_{t}^{1} (1-\tau) \quad (1-\tau+t)\, d\tau \\ &= \{\text{after polynomial development, integration, and factorization}\} \\ &= \frac{1}{3}(t^3 - 3t + 2) + \frac{1}{6}t(t^2 - 6t + 6), \quad (0 < t \leq 1) \end{aligned}$$

Case 5 : $t-1 \geq 0, \quad t-1 \leq 1$

As long as trailing point $(t-1)$ of $x(t-\tau)$ is in $\tau \in (0, 1)$, i.e.

$$\left.\begin{aligned} t-1 \geq 0 \quad &\therefore \quad t > 1 \\ t-1 < 1 \quad &\therefore \quad t < 2 \end{aligned}\right\} \quad \therefore \quad 1 \leq t \leq 2$$

the overlapping surface is bound by $(1-\tau)$ and $(1+\tau-t)$ linear segments, and $(t-1, 1)$ interval, see Fig. 3.56. Thus, product of these two linear segments defines convolution integral as:

$$\begin{aligned} z(t) &= \int_{t-1}^{1} (1-\tau) \quad (1+\tau-t)\, d\tau \\ &= \{\text{similar to the case 2}\} \\ &= -\frac{1}{6}(t-2)^3 \end{aligned}$$

Fig. 3.56 Example P.3.16

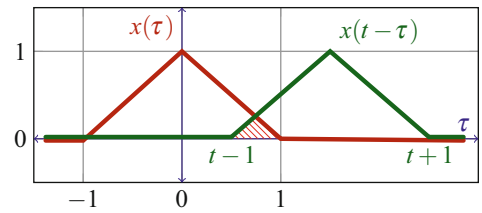
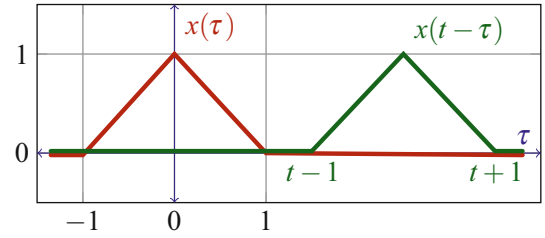


Fig. 3.57 Example P.3.16



Case 6 : $t-1>1$

The trailing point $(t-1)$ of $x(t-\tau)$ is superior to $\tau=1$, i.e.

$$t-1>1 \quad \therefore \quad t>2$$

It is irrelevant how much exactly t is superior to $\tau=1$, see Fig. 3.57. What is important is that under this condition at least one of the two functions equal zero, therefore their product equal zero as

$$x(\tau)\,x(t-\tau)=0, \quad \forall\tau$$

and by consequence, convolution integral equal zero as

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\tau)\; x(t-\tau)\; d\tau = \int_{-\infty}^{\infty} 0\; d\tau \\ &= 0 \quad (t \geq 2) \end{aligned}$$

Summary:

The total convolution integral is therefore the sum of these six cases, see also Fig. 3.58.

$$z(t)=\begin{cases} 0 & t<-2 \\ 1/6(t+2)^3 & -2\leq t\leq -1 \\ 1/3(-t^3+3t+2)-\frac{1}{6}t(t^2+6t+6) & -1\leq t\leq 0 \\ 1/3(t^3-3t+2)+\frac{1}{6}t(t^2-6t+6) & 0\leq t\leq 1 \\ -1/6(t-2)^3 & 1\leq t\leq 2 \\ 0 & t>2 \end{cases}$$

3.17. Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Preparatory work : $y(t)\rightarrow y(\tau)\rightarrow y(-\tau)\rightarrow y(t-\tau)$ transformations.

(a) $y(t)\rightarrow y(\tau)$ transformation does not change form, only the variable $t\rightarrow\tau$, Fig. 3.59 (top).
(b) $y(\tau)\rightarrow y(-\tau)$ time inversion transformation results in the mirrored form, Fig. 3.59 (middle). Note that $\tau=0$ reference point is still known.

Fig. 3.58 Example P.3.16

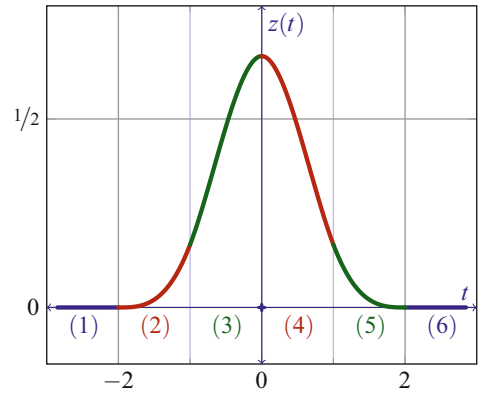
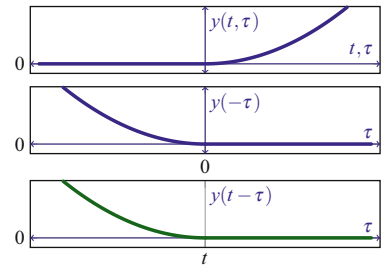


Fig. 3.59 Example P.3.17



(c) $y(-\tau) \rightarrow y(t-\tau)$ transformation shifts each point of $y(-\tau)$ by t, Fig. 3.59 (bottom). Note that $\tau = 0$ reference point is not known, it is relative to t, thus is resolved case by case.

Note that,

$$y(t) = t^2\, u(t)$$

$$\therefore$$

$$y(t-\tau) = (t-\tau)^2\, u(t-\tau) = (t^2 - 2t\,\tau + \tau^2)\, u(t-\tau)$$

There are two distinct cases to solve.

Case 1 : $t \leq 0$:

The leading t point of $y(t-\tau)$ is inferior to zero, i.e.

$$t \leq 0$$

It is irrelevant where exactly is $t < 0$ point, see Fig. 3.60. What is important is that under this condition, either $x(\tau)$ or $y(t-\tau)$ function equal zero, as

$$x(\tau)\; y(t-\tau) = 0, \quad \forall\tau$$

and by consequence convolution integral equal zero, as

Fig. 3.60 Example P.3.17

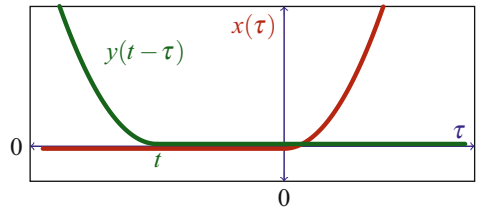
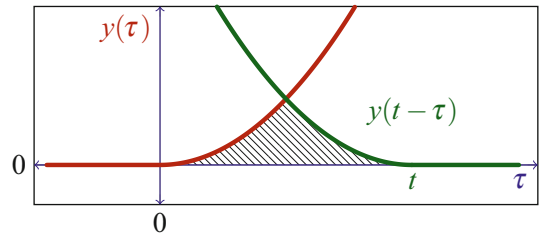


Fig. 3.61 Example P.3.17



$$z(t) = \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\ d\tau$$
$$= \int_{-\infty}^{\infty} 0\ d\tau = 0$$

Case 2 : $t > 0$:

The leading t point of $y(t-\tau)$ is superiour to zero, i.e.

$$t > 1$$

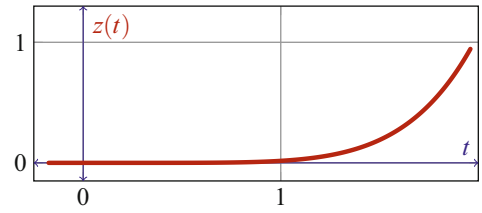
It is irrelevant where exactly is $t > 0$ point, see Fig. 3.61, as

$$z(t) = \int_{-\infty}^{\infty} x(\tau)\ y(t-\tau)\ d\tau = \int_{-\infty}^{\infty} \tau^3 u(\tau)\ (t-\tau)^2 u(t-\tau)\ d\tau$$
$$= \int_0^t \tau^3\ (t-\tau)^2\ d\tau = \int_0^t (\tau^5 - 2t\tau^4 + t^2\tau^3)\ d\tau$$
$$= \frac{1}{6}\tau^6\Big|_0^t - \frac{2t}{5}\tau^5\Big|_0^t + \frac{t^2}{4}\tau^4\Big|_0^t = \frac{t^6}{6} - \frac{2t^6}{5} + \frac{t^6}{4}$$
$$= \frac{t^6}{60}$$

Summary:

The total integral is therefore the sum of these two cases, see Fig. 3.62,

Fig. 3.62 Example P.3.17



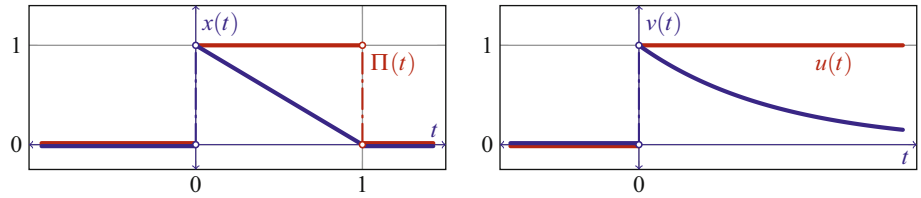
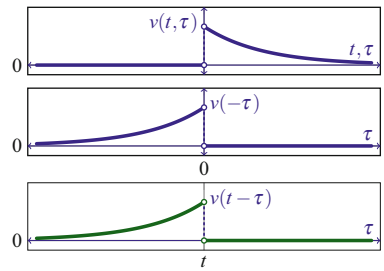


Fig. 3.63 Example P.3.18

Fig. 3.64 Example P.3.18



$$z(t) = \begin{cases} 0 & t \le 0 \\ \dfrac{t^6}{60} & t > 0 \end{cases} = \frac{t^6}{60}\, u(t)$$

3.18. Given analytical forms of two functions

$$x(t) = \begin{cases} 1-t; & 0 \le t \le 1 \\ 0; & \text{otherwise} \end{cases} \quad \text{and,} \quad v(t) = \begin{cases} e^{-t}; & t \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

where there are well defined boundaries, these two functions $x(t)$ and $v(t)$ may also be defined by products with step and port functions, see Fig. 3.63.

$$x(t) = (1-t)\ \Pi(t) \quad \text{and} \quad v(t) = e^{-t}\ u(t)$$

Graphical method of resolving convolution integral consist the following steps, where the original integral form is decomposed into several simpler integrals.

Preparatory work : $v(t) \to v(\tau) \to v(-\tau) \to v(t-\tau)$ transformations.

(a) $v(t) \to v(\tau)$ transformation does not change form, only the variable $t \to \tau$, Fig. 3.64 (top).

$v(\tau) \rightarrow v(-\tau)$ time inversion transformation results in the mirrored form, Fig. 3.64 (top). Note that $\tau = 0$ reference point is still known.

(b) $v(-\tau) \rightarrow v(t-\tau)$ transformation shifts each point of $v(-\tau)$ by t, Fig. 3.64 (bottom). Note that $\tau = 0$ reference point is not known, it is relative to t, thus is resolved case by case.

There are three distinct cases to be solved.

Case 1 : $t < 0$

Within this interval, either one of the two function equal zero, see Fig. 3.65, thus their product equal zero, as

$$x(\tau)\ v(t-\tau) = 0, \quad \forall\tau$$

and by consequence convolution integral equal zero, as

$$w(t) = \int_{-\infty}^{\infty} x(\tau)\ v(t-\tau)\ d\tau = \int_{0}^{t-1} (0)\ (0)\ d\tau$$
$$= 0 \quad (t < 0)$$

Case 2 : $0 \le t \le 1$:

The leading edge point t of $v(t-\tau)$ is found in interval $\tau \in (0, 1)$, i.e.

$$\left.\begin{array}{ll} t & \ge 0 \\ t & \le 1 \end{array}\right\} \quad \therefore \quad 0 \le t \le 1$$

where the overlapping surface is bound by $\tau \in (0, t)$ inteval, see Fig. 3.66, as

$$w(t) = \int_{-\infty}^{\infty} x(\tau)\ v(t-\tau)\ d\tau = \int_{0}^{t} (1-\tau)\ e^{-(t-\tau)}\ d\tau = \int_{0}^{t} \left(e^{-t}e^{\tau} - \tau\, e^{-t}e^{\tau}\right)\ d\tau$$
$$= e^{-t}\left[\int_{0}^{t} e^{\tau}\ d\tau - \int_{0}^{t} \tau\, e^{\tau}\ d\tau\right] = e^{-t}\left[I_1 - I_2\right]$$

Where,

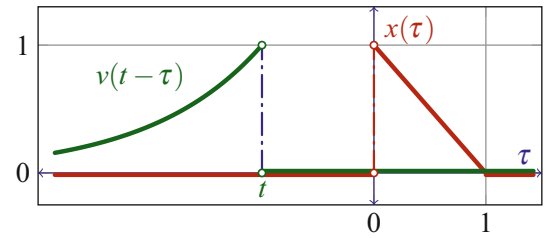


Fig. 3.65 Example P.3.18

Fig. 3.66 Example P.3.18

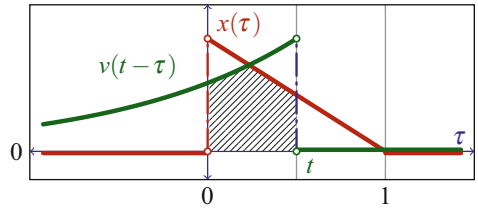
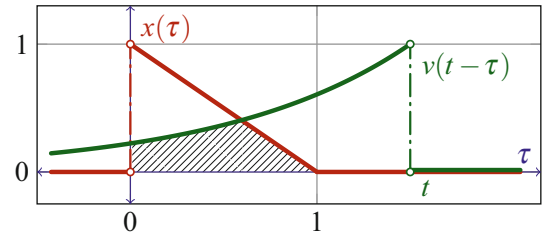


Fig. 3.67 Example P.3.18



$$I_1 = \int_0^t e^{\tau}\, d\tau = e^{\tau}\Big|_0^t = e^t - 1$$

and,

$$I_2 = \int_0^t \tau\, e^{\tau}\, d\tau = \{\text{partial integration}\} = t\, e^t - e^t + 1$$

Therefore,

$$\begin{aligned} w(t) &= e^{-t}\left(e^t - 1 - t\, e^t + e^t - 1\right) \\ &= 2 - t - 2\, e^{-t} \quad (0 \le t \le 1) \end{aligned}$$

Case 3 : $t > 1$

The leading edge of $v(t-\tau)$ is in $(t > 1)$ interval, thus the overlapping surface is bound in $\tau \in (0, 1)$ interval, see Fig. 3.67, as

$$\begin{aligned} w(t) &= \int_{-\infty}^{\infty} x(\tau)\; v(t-\tau)\, d\tau = \int_0^1 (1-\tau)\; e^{-(t-\tau)}\, d\tau \\ &= e^{-t} \int_0^1 (e^{\tau} - \tau\, e^{\tau})\; d\tau = \{\text{etc. as in Case 2. }\} \\ &= e^{-t}(e-2) \quad (t > 1) \end{aligned}$$

Summary:

The total convolution integral is therefore, see Fig. 3.68,

Fig. 3.68 Example P.3.18

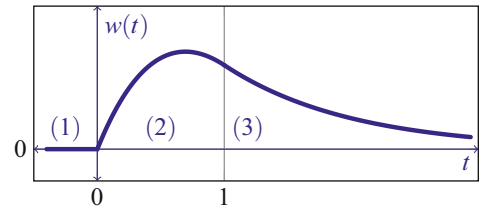
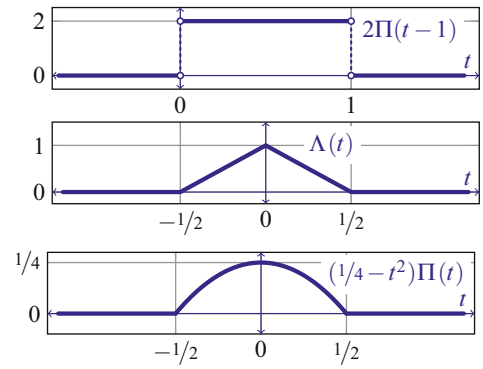


Fig. 3.69 Example P.3.19



$$w(t) = \begin{cases} 0; & t < 0 \\ 2 - t - 2\,e^{-t}; & 0 \leq t \leq 1 \\ e^{-t}(e-2); & t > 1 \end{cases}$$

3.4 Energy and Power

3.19. Amplitude of a delayed port signal $x(t) = 2\Pi(t-1)$ equal two for $t \in (0,1)$, otherwise it equal zero. Thus,

$$E_x \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |2\Pi(t-1)|^2\,dt = \int_0^1 (2)^2\,dt = 4$$

Therefore, given a finite source of energy,

$$\langle P_x \rangle \overset{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = \lim_{t\to\infty} \frac{4}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals that are contained within a finite time interval (t_1, t_2) and whose amplitude is finite, such as three forms in Fig. 3.69. By consequence, $E_x < \infty$ and $\langle P_x \rangle = 0$.

3.20. Triangular signal $x(t) = \Lambda(t)$ is bound in $t \in (-1/2, 1/2)$ interval, thus

$$\begin{aligned} E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\, dt &= \int_{-\infty}^{\infty} |\Lambda(t)|^2\, dt = \int_{-1/2}^{0} (t+1/2)^2\, dt + \int_{0}^{1/2} (-t+1/2)^2\, dt \\ &= \frac{1}{3}\, t^3 \Big|_{-1/2}^{0} + \frac{1}{2}\, t^2 \Big|_{-1/2}^{0} + \frac{1}{4}\, t \Big|_{-1/2}^{0} + \frac{1}{3}\, t^3 \Big|_{0}^{1/2} - \frac{1}{2}\, t^2 \Big|_{0}^{1/2} + \frac{1}{4}\, t \Big|_{0}^{1/2} \\ &= \frac{1}{24} + \frac{1}{24} = \frac{1}{12} \end{aligned}$$

Therefore, given a finite source of energy,

$$\langle P_x \rangle \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = \lim_{t\to\infty} \frac{1/12}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals that are contained within a finite time interval (t_1, t_2) and whose amplitude is finite, see Fig. 3.69. By consequence, $E_x < \infty$ and $\langle P_x \rangle = 0$.

3.21. Non–zero part of quadratic signal $x(t) = \left(1/4 - t^2\right)\, \Pi(t)$ is bound in $t \in (-1/2, 1/2)$ interval, and its magnitude is finite. Thus, non–zero integrals are

$$\begin{aligned} E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\, dt &= \int_{-\infty}^{\infty} |\left(1/4 - t^2\right)\, \Pi(t)|^2\, dt = \int_{-1/2}^{1/2} |\left(1/4 - t^2\right)\, (1)|^2\, dt \\ &= \frac{1}{5}\, x^5 \Big|_{-1/2}^{1/2} - \frac{1}{6}\, x^3 \Big|_{-1/2}^{1/2} + \frac{1}{16}\, x \Big|_{-1/2}^{1/2} = \frac{1}{30} \end{aligned}$$

Therefore, given a finite source of energy,

$$\langle P_x \rangle \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = \lim_{t\to\infty} \frac{1/30}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals that are contained within a finite time interval (t_1, t_2) and whose amplitude is finite, see Fig. 3.69. By consequence, $E_x < \infty$ and $\langle P_x \rangle = 0$.

3.22. Given a continuous non–periodic infinitely long signal, its average power is calculated as

$$\begin{aligned} \langle P(t) \rangle &\stackrel{\text{def}}{=} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |a|^2\, dt = a^2 \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \\ &= a^2 \lim_{\tau\to\infty} \frac{1}{\tau}\, t \Big|_{-\tau/2}^{\tau/2} = a^2 \lim_{\tau\to\infty} \frac{1}{\tau} \left(\frac{\tau}{2} - \frac{-\tau}{2}\right) = a^2 \lim_{\tau\to\infty} \frac{\tau}{\tau} = a^2 \end{aligned}$$

And,

$$\begin{aligned} E(t) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |a|^2 \, dt = a^2 \int_{-\infty}^{\infty} dt \\ &= a^2 \, t \Big|_{-\infty}^{\infty} = a^2 \, (\infty + \infty) \\ &= \infty \end{aligned}$$

That is to say, in order to sustain the non-zero finite average power over infinitely long period of time, evidently, it is necessary to have infinite source of energy.

Summary : this signal $x(t)$ belongs to the category of signals that are *not* contained within a finite time interval and whose amplitude is constant, such as forms in Fig. 3.70. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.23. Given an infinitely long $u(t)$ signal, its energy is

$$E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |u(t)|^2 \, dt = \int_{0}^{\infty} (1)^2 \, dt = t \Big|_{0}^{\infty} = \infty$$

And its average power is

$$\begin{aligned} \langle P_x \rangle &\stackrel{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |u(t)|^2 \, dt \\ &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{0} |0|^2 \, dt + \lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau/2} |1|^2 \, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \, t \Big|_{0}^{\tau/2} = \lim_{\tau \to \infty} \frac{1}{\cancel{\tau}} \frac{\cancel{\tau}}{2} \\ &= \frac{1}{2} \end{aligned}$$

which makes sense because half of the time $u(t)$ signal amplitude equals zero (for $t < 0$) and the other half of time (i..e. $t \geq 0$) it equal one, thus average of '0' and '1' is one half.

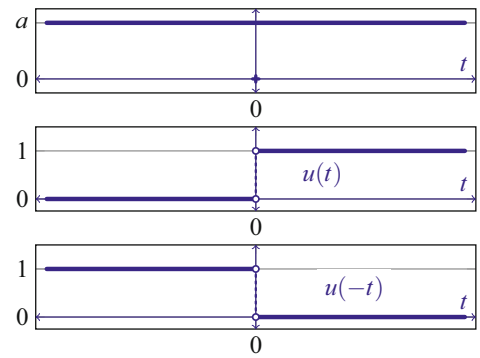


Fig. 3.70 Example P.3.22

That is to say, in order to sustain the non-zero finite average power over infinitely long period of time, evidently, it is necessary to have infinite source of energy.

Summary : this signal $x(t)$ belongs to the category of signals that are *not* bound within a finite time interval and whose amplitude is constant, see Fig. 3.70. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.24. Given an infinitely long $u(-t)$ signal, its energy is

$$E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |u(-t)|^2 \, dt = \int_{-\infty}^{0} (1)^2 \, dt = t \Big|_{-\infty}^{0} = \infty$$

And its average power is

$$\begin{aligned}
\langle P_x \rangle &\stackrel{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |u(-t)|^2 \, dt \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{0} |1|^2 \, dt + \lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau/2} |0|^2 \, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \, t \Big|_{-\tau/2}^{0} = \lim_{\tau \to \infty} \frac{1}{\cancel{\tau}} \frac{\cancel{\tau}}{2} \\
&= \frac{1}{2}
\end{aligned}$$

which makes sense because half of the time $u(t)$ signal amplitude equals zero (for $t < 0$) and the other half of time (i..e. $t \geq 0$) it equal one, thus average of '0' and '1' is one half.

That is to say, in order to sustain the non–zero finite average power over infinitely long period of time, evidently, it is necessary to have infinite source of energy.

Summary : this signal $x(t)$ belongs to the category of signals that are *not* bound within a finite time interval and whose amplitude is constant, see Fig. 3.70. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.25. Given a continuous periodic infinitely long signal its average power over one period T is

$$\begin{aligned}
\langle P_x \rangle &\stackrel{\text{def}}{=} \frac{1}{T} \int_{-T/2}^{T/2} |\sin(t)|^2 \, dt = \frac{1}{T} \int_{-T/2}^{T/2} \sin^2(t) \, dt = \left\{ \sin^2(x) = \frac{1 - \cos 2x}{2} \right\} \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1 - \cos 2x}{2} \right) dt = \frac{1}{T} \left[\frac{1}{2} t - \frac{1}{4} \sin(2t) \right]_{-T/2}^{T/2} \\
&= \frac{1}{T} \left[\frac{1}{2} \left(\frac{T}{2} + \frac{T}{2} \right) - \frac{1}{4} \left(\sin\left(2 \frac{T}{2} \right) - \sin\left(-2 \frac{T}{2} \right) \right) \right] \\
&= \frac{1}{\cancel{T}} \left[\frac{1}{2} \cancel{T} - \frac{1}{4} \left(\cancel{\sin(T)} \to 0 + \cancel{\sin(T)} \to 0 \right) \right] \\
&= \frac{1}{2}
\end{aligned}$$

That is to say, average power within a singe period is non–zero. By consequence, in order to sustain the same non–zero average power over infinitely many periods there must be an infinite source of energy, as

$$E_x \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle P_x \rangle \; t = \frac{1}{2} \lim_{t\to\infty} \; t = \infty$$

Summary : this signal $x(t)$ belongs to the category of *periodic* signals, i.e. that are *not* bound within a finite time interval and whose amplitude is finite. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.26. Given a complex exponential signal, by Euler's formula it is converted as

$$x(t) = a\, e^{j\,\omega_0\, t} = a\,[\cos(\omega_0\, t) - j\,\sin(\omega_0\, t)] \quad \therefore \quad \omega_0 T_0 = 2\pi \quad \therefore \quad T_0 = \frac{2\pi}{\omega_0}$$

which is to say that period of $x(t)$ is T_0. Average power of a periodic signal is calculated as

$$\begin{aligned}\langle P_x \rangle &\stackrel{\text{def}}{=} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |a\, e^{j\,\omega_0\, t}|^2 \; dt = \left\{ \left| e^{jx} \right| = 1 \right\} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} a^2 \, \cancelto{1}{|e^{j\,\omega_0\, t}|^2} \; dt \\ &= \frac{a^2}{T_0} \; t \; \Big|_{-T_0/2}^{T_0/2} = \frac{a^2}{T_0} \left(\frac{T_0}{2} + \frac{T_0}{2} \right) = a^2\end{aligned}$$

This is periodic signal, therefore,

$$E_x \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle P_x \rangle \, t = \infty$$

That is to say, in order to sustain a non–zero average power over infinitely many periods it must be that $E_x = \infty$.

Summary : this signal $x(t)$ belongs to the category of *periodic* signals, i.e. that are *not* bound within a finite time interval and whose amplitude is finite. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.27. Given a continuous periodic infinitely long signal its average power over one period T is

$$\begin{aligned}\langle P_x \rangle &\stackrel{\text{def}}{=} \frac{1}{T} \int_{-T/2}^{T/2} |1 + \cos(2t)|^2 \; dt = \frac{1}{T} \int_{-T/2}^{T/2} \left(\cos^2(2t) + 2\cos(2t) + 1 \right) \; dt \\ &= \left\{ \cos^2(x) = \frac{1 + \cos 2x}{2} \right\} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1}{2} + \frac{1}{2}\cos(4t) + 2\cos(2t) + 1 \right) \; dt \\ &= \frac{1}{T} \left[\frac{1}{8} \; \sin(4t) + \sin(2t) + \frac{3}{2}\, t \right]_{-T/2}^{T/2}\end{aligned}$$

$$= \frac{1}{T}\left[\frac{1}{8}\sin\left(4\,2\,\frac{T}{2}\right) + \sin\left(2\frac{T}{2}\right) + \frac{3}{2}\left(\frac{T}{2}+\frac{T}{2}\right)\right]$$
$$= \frac{1}{T}\left[\frac{1}{8}\underbrace{\sin(2T)}_{\to 0} + \underbrace{\sin(T)}_{\to 0} + \frac{3}{2}T\right]$$
$$= \frac{3}{2}$$

That is to say, average power within a singe period is non–zero. By consequence, in order to sustain the same non–zero average power over infinitely many periods there must be an infinite source of energy, as

$$E_x \overset{\text{def}}{=} \lim_{t\to\infty} \langle P_x \rangle\ t = \frac{3}{2}\ \lim_{t\to\infty}\ t = \infty$$

Summary : this signal $x(t)$ belongs to the category of *periodic* signals, i.e. that are *not* bound within a finite time interval and whose amplitude is finite. By consequence, $E_x = \infty$ and $0 < \langle P_x \rangle < \infty$.

3.28. Given exponential signal converges, i.e. $x(t) \to 0$ for $t \to \pm\infty$, thus

$$E_x \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\ dt = \int_{-\infty}^{\infty} \left|e^{-t}\ u(t)\right|^2\ dt = \int_0^{\infty} e^{-2t}\ (1)^2\ dt = -\frac{1}{2}\ e^{-2t}\ \Big|_0^{\infty}$$
$$= -\frac{1}{2}\,[0-1] = \frac{1}{2}$$

Therefore,

$$\langle P_x \rangle \overset{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \pm\infty)$ then $(x(t) \to 0)$, such as forms in Fig. 3.71. By consequence, $0 < E_x < \infty$ and $\langle P_x \rangle = 0$.

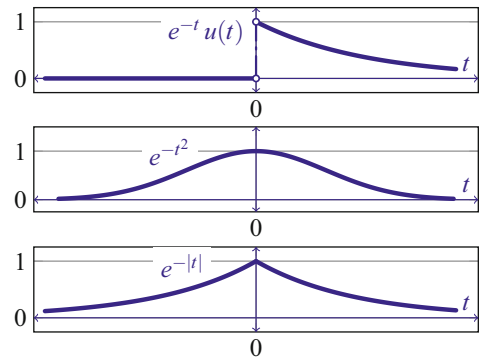


Fig. 3.71 Example P.3.28

3.29. Given exponential signal converges, i.e. $x(t) \to 0$ for $t \to \pm\infty$, thus

$$E_x \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left|e^{-t^2}\right|^2 \, dt = \int_{-\infty}^{\infty} \left(e^{-t^2}\right)^2 \, dt = \{\text{see chapters on integrals}\} = \sqrt{\frac{\pi}{2}}$$

Therefore,

$$\langle P_x \rangle \overset{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \pm\infty)$ then $(x(t) \to 0)$, see Fig. 3.71. By consequence, $0 < E_x < \infty$ and $\langle P_x \rangle = 0$.

3.30. Given exponential signal converges, i.e. $x(t) \to 0$ for $t \to \pm\infty$, thus

$$E_x \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left|e^{-|t|}\right|^2 \, dt = \int_{-\infty}^{0} \left(e^{t}\right)^2 \, dt + \int_{0}^{\infty} \left(e^{-t}\right)^2 \, dt = \{\text{see A.3.28}\}$$

$$= \frac{1}{2} e^{2t} \Big|_{-\infty}^{0} + \frac{1}{2} e^{-2t} \Big|_{0}^{\infty} = 1$$

Therefore,

$$\langle P_x \rangle \overset{\text{def}}{=} \lim_{t\to\infty} \frac{E_x}{t} = 0$$

That is to say, having a finite source of energy that must be distributed over infinitely long period of time, the average power reduces to zero.

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \pm\infty)$ then $(x(t) \to 0)$, see Fig. 3.71. By consequence, $0 < E_x < \infty$ and $\langle P_x \rangle = 0$.

3.31. Given an $r(t)$ signal, its energy is

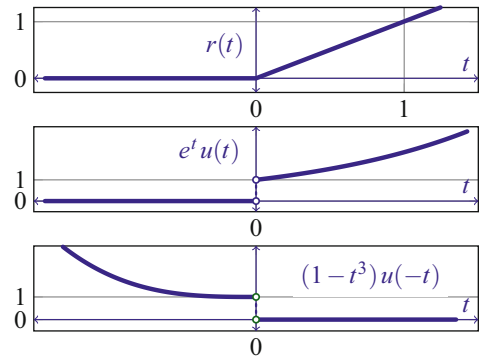
$$E_x \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |r(t)|^2 \, dt = \int_{0}^{\infty} r^2 \, dt = \frac{1}{3} \, t^3 \Big|_{0}^{\infty} = \infty$$

And its average power is

$$\langle P_x \rangle \overset{\text{def}}{=} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |r(t)|^2 \, dt$$

$$= 0 + \lim_{\tau\to\infty} \frac{1}{\tau} \int_{0}^{\tau/2} r^2 \, dt = \frac{1}{3} \lim_{\tau\to\infty} \frac{1}{\tau} \, t^3 \Big|_{0}^{\tau/2} = \frac{1}{3} \lim_{\tau\to\infty} \frac{1}{\tau} \frac{\tau^3}{8}$$

Fig. 3.72 Example P.3.31



$$= \frac{1}{3} \lim_{\tau \to \infty} \frac{\tau^2}{8} = \infty$$

This is an example of a signal whose both energy and average power equal infinity.

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \infty)$ or $(t \to -\infty)$ then $(x(t) \to \infty)$, such as forms in Fig. 3.72. By consequence, $E_x = \infty$ and $\langle P_x \rangle = \infty$.

3.32. Given exponential $e^t\, u(t)$ signal, its energy is

$$E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |e^t\, u(t)|^2\, dt = \int_0^{\infty} e^{2t}\, (1)^2\, dt = \frac{1}{2}\, e^{2t} \Big|_0^{\infty} = \infty$$

$$\langle P_x \rangle \stackrel{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |e^t\, u(t)|^2\, dt$$

$$= 0 + \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau/2} e^{2t}\, dt = \frac{1}{2} \lim_{\tau \to \infty} \frac{1}{\tau}\, e^{2t} \Big|_0^{\tau/2} = \frac{1}{2} \lim_{\tau \to \infty} \frac{1}{\tau}\, (e^{\tau} - 1)$$

$$= \frac{1}{2} \lim_{\tau \to \infty} \frac{e^{\tau}}{\tau}^{\to \infty} - \frac{1}{2} \lim_{\tau \to \infty} \frac{1}{\tau}^{\to 0} = \infty$$

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \infty)$ or $(t \to -\infty)$ then $(x(t) \to \infty)$, see Fig. 3.72. By consequence, $E_x = \infty$ and $\langle P_x \rangle = \infty$.

3.33. Given cubic $x(t)$ signal, its energy is

$$E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |(1 - x^3)\, u(-t)|^2\, dt = \int_{-\infty}^{0} (1 - x^3)^2\, (1)^2\, dt$$

$$= \frac{1}{7}\, t^7 \Big|_{-\infty}^{0} - \frac{1}{2}\, t^4 \Big|_{-\infty}^{0} + t \Big|_{-\infty}^{0} = \infty$$

And its average power is

$$\langle P_x \rangle \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \left|(1-x^3)\, u(-t)\right|^2 \, dt$$

$$= \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{0} \left(1-x^3\right)^2 \, dt + 0 = \frac{1}{7} \lim_{\tau\to\infty} \frac{1}{\tau} \, t^7 \Big|_{-\tau/2}^{0} - \frac{1}{2} \lim_{\tau\to\infty} \frac{1}{\tau} \, t^4 \Big|_{-\tau/2}^{0} + \lim_{\tau\to\infty} \frac{1}{\tau} \, t \Big|_{-\tau/2}^{0}$$

$$= \infty$$

Summary : this signal $x(t)$ belongs to the category of signals where if $(t \to \infty)$ or $(t \to -\infty)$ then $(x(t) \to \infty)$, see Fig. 3.72. By consequence, $E_x = \infty$ and $\langle P_x \rangle = \infty$.

Discrete Time Convolution

4

Convolution sum is defined as

$$y[n] = x[k] * y[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \tag{4.1}$$

where k is the provisional summation variable.

Energy and power definitions: by definition, $E(t) = \int_0^t P(\tau)d\tau$ for continuous functions. Calculations of sampled functions are the following:

1. Energy within an interval

$$E_x = \sum_{k=n}^{m} |x[k]|^2$$

2. Total energy

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2$$

3. Average power within an interval

$$\langle P_x \rangle = \frac{1}{2n+1} \sum_{k=n}^{n} |x[k]|^2$$

4. Total power

$$P_x = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=n}^{n} |x[k]|^2$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_4

Problems

4.1 Discrete Signal Convolution

Given short sequences in P.4.1 to P.4.3, calculate their convolution sums.

4.1. $x[n] = [\underline{1}, 1, 1]$, $y[n] = [\underline{1}, 1, 1]$

4.2. $x[n] = [1, 2, 3]$, $y[n] = [1, 1, 1]$

4.3. $f[k]$, $g[k]$, see Fig. 4.1

Given two short sequences $h[n] = [1, \underline{2}, 0, -3]$ and $x[n]$ in P.4.4 to P.4.7, calculate their convolution sums. (Note: number whose index $n = 0$ is underlined.)

4.4. $x[n] = \delta[n]$

4.5. $x[n] = \delta[n+1] + \delta[n-2]$

4.6. $x[n] = [\underline{1}, 1, 1]$

4.7. $x[n] = [2, 1, \underline{-1}, -2, -3]$

Given analytical form of sequences in P.4.8 to P.4.10, calculate their convolution sums.

4.8. $x[k] = \alpha^k\, u[k]; \quad (0 < \alpha < 1)$

$h[k] = u[k]$

4.9. $x[k] = \begin{cases} 1; & (0 \le k \le 4) \\ 0; & \text{otherwise} \end{cases}$

$h[k] = \begin{cases} \alpha^k; & (0 \le k \le 6) \\ 0; & \text{otherwise} \end{cases}$

4.10. $x[n] = \sin(\omega_0 n)\, u(n)$

$y[n] = u[n]$

4.2 Discrete Signal Energy and Power

Calculate energy and power of sequences $x[k]$ in P.4.11 to P.4.16.

4.11. $x[k] = \delta[k]$

4.12. $x[k] = \left(\frac{1}{4}\right)^k u[k]$

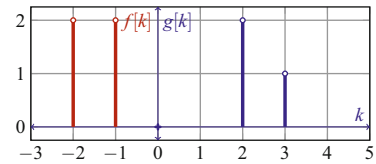
4.13. $x[k] = e^{j\,10k}\, u[k]$

4.14. $x[k] = \begin{cases} \alpha^k\,; & k \ge 0 \\ 0\,; & k < 0 \end{cases}$

4.15. $x[k] = u[k]$

4.16. $x[k] = \cos\left[\frac{\pi}{6} k\right]$

Fig. 4.1 Example P.4.3



Answers

4.1 Discrete Signal Convolution

There are many ways to tabulate convolution sums, mostly the question of preference and style. Some of the typical methods are illustrated here.

4.1. There is a number of ways to tabulate convolution of discrete series, which is question of style and preference.

Method 1: Given two short sequences $x[n] = [\underline{1}, 1, 1]$ and $y[n] = [\underline{1}, 1, 1]$, that is to say with a clearly defined finite interval of non–zero samples, the interval of convolution $z[n]$ series is determined as

$$
\begin{aligned}
x[0] = 1, x[1] = 1, x[2] = 1 \quad &\therefore \quad \text{finite non–zero interval begins at} \quad n = 0 \quad \text{and ends at} \quad n = 2 \\
y[0] = 1, y[1] = 1, y[2] = 1 \quad &\therefore \quad \text{finite non–zero interval begins at} \quad n = 0 \quad \text{and ends at} \quad n = 2 \\
&\therefore \\
z[n] \quad &\therefore \quad \text{finite non–zero interval begins at} \quad n = 0 + 0 = 0 \\
&\therefore \quad \text{finite non–zero interval ends at} \quad n = 2 + 2 = 4
\end{aligned}
$$

that is to say, $z[n], \ 0 \leq n \leq 4$. By consequence, it is necessary to evaluate $z[n]$ in five cases $n = [0, 1, 2, 3, 4]$, as

$$
z[n] = x[n] * y[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} x[m]\, y[k-m] = \sum_{m=0}^{2} x[m]\, y[n-m]
$$

Note that "fixed" series is $x[m]$ and $y[k-m]$ is the "sliding" sequence relative to parameter m, therefore there are only three simples in the interval $m = (0, -1)$ as,

$$
\begin{aligned}
\underline{\textit{Case 1: } n = 0} \quad z[\,0] &= \sum_{m=0}^{2} x[m]\, y[\,0-m] = x[0]\, y[\,0-(0)] + x[1]\, y[\,0-(1)] + x[2]\, y[\,0-(2)] \\
&= x[0]\, y[0] + x[1]\, y[-1] + x[2]\, y[-2] = (1)(1) + (1)(0) + (1)(0) \\
&= 1 \\
\underline{\textit{Case 2: } n = 0} \quad z[\,1] &= \sum_{m=0}^{2} x[m]\, y[\,1-m] = x[0]\, y[\,1-(0)] + x[1]\, y[\,1-(1)] + x[2]\, y[\,1-(2)] \\
&= x[0]\, y[1] + x[1]\, y[0] + x[2]\, y[-1] = (1)(1) + (1)(1) + (1)(0) \\
&= 2
\end{aligned}
$$

$$\underline{\textit{Case 3: } n=0} \quad z[2] = \sum_{m=0}^{2} x[m]\, y[2-m] = x[0]\, y[2-(0)] + x[1]\, y[2-(1)] + x[2]\, y[2-(2)]$$
$$= x[0]\, y[2] + x[1]\, y[1] + x[2]\, y[0] = (1)(1) + (1)(1) + (1)(1)$$
$$= 3$$

$$\underline{\textit{Case 4: } n=0} \quad z[3] = \sum_{m=0}^{2} x[m]\, y[3-m] = x[0]\, y[3-(0)] + x[1]\, y[3-(1)] + x[2]\, y[3-(2)]$$
$$= x[0]\, y[3] + x[1]\, y[2] + x[2]\, y[1] = (1)(0) + (1)(1) + (1)(1)$$
$$= 2$$

$$\underline{\textit{Case 5: } n=0} \quad z[4] = \sum_{m=0}^{2} x[m]\, y[4-m] = x[0]\, y[4-(0)] + x[1]\, y[4-(1)] + x[2]\, y[4-(2)]$$
$$= x[0]\, y[4] + x[1]\, y[3] + x[2]\, y[2] = (1)(0) + (1)(0) + (1)(1)$$
$$= 1$$

In conclusion, $z[n] = [\underline{1}, 2, 3, 2, 1]$, see Fig. 4.2. (Compare with the continuous time example P.3.12). Note that, if given as a list, sample $z[0]$ is underlined.

Method 2: Given two short sequences $x[n] = [\underline{1}, 1, 1]$ and $y[n] = [\underline{1}, 1, 1]$, sequences $x[n]$ may be written, for example, as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

By definition, convolution sum is

$$z[n] = x[n] * y[n] = (\delta[n] + \delta[n-1] + \delta[n-2]) * y[n] = y[n] + y[n-1] + y[n-2]$$

where, $y[n-1]$ and $y[n-2]$ are shifted $y[n]$ as

$$y[n-1 = [_, 1, 1, 1]$$
$$y[n-2] = [_, , 1, 1, 1]$$

so that '_' syntax is used to mark position of $n = 0$ sample. The convolution sum

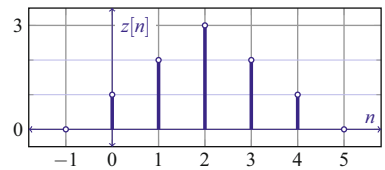


Fig. 4.2 Example P.4.1

$$z[n] = y[n] + y[n-1] + y[n-2]$$

may be done by the tabular method as

n		0	1	2	3	4
$y[n]$		1	1	1		
$y[n-1]$			1	1	1	
$y[n-2]$	+			1	1	1
$z[n]$		1	2	3	2	1

which is in agreement with the **Method 1:** result.

4.2. There is a number of ways to tabulate convolution of discrete series, which is question of style and preference.

Method 1: Given two short sequences $f[k]$ and $g[k]$ in Fig. 4.1, that is to say with a clearly defined finite interval of non–zero samples, the interval of convolution $h[k]$ series is determined as

$$\begin{aligned}
f[-2] = 2,\ f[-1] = 2 \quad &\therefore \quad \text{finite non–zero interval begins at} \quad k = -2 \quad \text{and ends at} \quad k = -1 \\
g[1] = 2,\ f[3] = 1 \quad &\therefore \quad \text{finite non–zero interval begins at} \quad k = 2 \quad \text{and ends at} \quad k = 3 \\
&\therefore \\
h[k] \quad &\therefore \quad \text{finite non–zero interval begins at} \quad k = -2 + 2 = 0 \\
&\therefore \quad \text{finite non–zero interval ends at} \quad k = -1 + 3 = 2
\end{aligned}$$

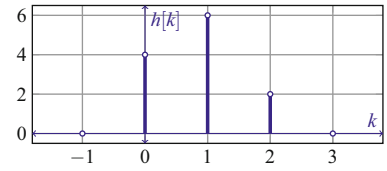
that is to say, $h[k] = 0, \quad 0 \le k \le 2$. By consequence, it is necessary to evaluate $h[k]$ in three cases $k = [0, 1, 2]$, as

$$h[k] = f[k] * g[k] \overset{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m]\, g[k-m] = \sum_{m=-2}^{-1} f[m]\, g[k-m]$$

Note that "fixed" series is $f[m]$ and $g[k-m]$ is the "sliding" sequence relative to parameter m, therefore there are only two simples in the interval $m = (-2, -1)$ as,

$$\begin{aligned}
\textit{Case 1: } k = 0 \quad h[0] &= \sum_{m=-2}^{-1} f[m]\, g[0-m] = f[-2]\, g[0-(-2)] + f[-1]\, g[0-(-1)] \\
&= f[-2]\, g[2] + f[-1]\, g[1] = (2)(2) + (2)(0) = 4 \\
\textit{Case 2: } k = 1 \quad h[1] &= \sum_{m=-2}^{-1} f[m]\, g[1-m] = f[-2]\, g[1-(-2)] + f[-1]\, g[1-(-1)] \\
&= f[-2]\, g[3] + f[-1]\, g[2] = (2)(1) + (2)(2) = 6
\end{aligned}$$

Fig. 4.3 Example P.4.3



$$\underline{\textit{Case 3: } k=2} \quad h[\,2] = \sum_{m=-2}^{-1} f[m]\; g[\,2-m] = f[-2]\; g[2-(-2)] + f[-1]\; g[2-(-1)]$$

$$= f[-2]\; g[4] + f[-1]\; g[3] = (2)(0) + (2)(1) = 2$$

In conclusion, $h[k] = [\underline{4}\,, 6, 2]$, see Fig. 4.3. Note that, if given as a list, sample $h[0]$ is underlined.

Method 2: Given graphs in Fig. 4.1, two sequences $f[k]$ and $g[k]$ may be written, for example, as

$$f[k] = 2\delta[k+2] + 2\delta[k+1]$$
$$g[k] = [\underline{\ }\,, , 2, 1]$$

where the leading empty spaces in $g[k]$ keep the sequence index so that $g[2] = 2$ and $g[3] = 1$. Underlined space '$\underline{\ }$' is used to indicate the index 0 position. By definition, convolution sum is

$$h[k] = f[k] * g[k] = (2\delta[k+2] + 2\delta[k+1]) * g[k] = 2\,g[k+2] + 2\,g[k+1]$$

where, $g[k+2]$ and $g[k+1]$ are shifted $g[k]$ as

$$2\,g[k+2] = 2\,[\underline{2}\,, 1] = [\underline{4}\,, 2]$$
$$2\,g[k+1] = 2\,[\underline{\ }\,, 2, 1] = [\underline{\ }\,, 4, 2]$$

so that $\underline{2}$ syntax is used to mark $k = 0$ index in $g[k+2]$ sequence. The convolution sum $h[k] = 2\,g[k+2] + 2\,g[k+1]$ may be done by the tabular method as

k		-2	-1	$\underline{0}$	1	2	3
$2\,g[k+2]$				$\underline{4}$	2		
$2\,g[k+1]$	+				4	2	
$h[k]$				$\underline{4}$	6	2	

which is in agreement with the **Method 1:** result.

4.3. There is a number of ways to tabulate convolution of discrete series, which is question of style and preference.

Method 1: Given two short sequences $f[k]$ and $g[k]$ in Fig. 4.1, that is to say with a clearly defined finite interval of non–zero samples, the interval of convolution $h[k]$ series is determined as

$$
\begin{aligned}
f[-2]=2,\, f[-1]=2 \;\; &\therefore \quad \text{finite non–zero interval begins at} \quad k=-2 \quad \text{and ends at} \quad k=-1 \\
g[1]=2,\, f[3]=1 \;\; &\therefore \quad \text{finite non–zero interval begins at} \quad k=2 \quad \text{and ends at} \quad k=3 \\
&\therefore \\
h[k] \;\; &\therefore \quad \text{finite non–zero interval begins at} \quad k=-2+2=0 \\
&\therefore \quad \text{finite non–zero interval ends at} \quad k=-1+3=2
\end{aligned}
$$

that is to say, $h[k]=0, \;\; 0 \le k \le 2$. By consequence, it is necessary to evaluate $h[k]$ in three cases $k=[0,\,1,\,2]$, as

$$h[k]=f[k]*g[k] \overset{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m]\, g[k-m] = \sum_{m=-2}^{-1} f[m]\, g[k-m]$$

Note that "fixed" series is $f[m]$ and $g[k-m]$ is the "sliding" sequence relative to parameter m, therefore there are only two simples in the interval $m=(-2,-1)$ as,

$$
\begin{aligned}
\underline{\textit{Case 1: } k=0} \quad h[0] &= \sum_{m=-2}^{-1} f[m]\, g[0-m] = f[-2]\, g[0-(-2)] + f[-1]\, g[0-(-1)] \\
&= f[-2]\, g[2] + f[-1]\, g[1] = (2)(2)+(2)(0) = 4 \\
\underline{\textit{Case 2: } k=1} \quad h[1] &= \sum_{m=-2}^{-1} f[m]\, g[1-m] = f[-2]\, g[1-(-2)] + f[-1]\, g[1-(-1)] \\
&= f[-2]\, g[3] + f[-1]\, g[2] = (2)(1)+(2)(2) = 6 \\
\underline{\textit{Case 3: } k=2} \quad h[2] &= \sum_{m=-2}^{-1} f[m]\, g[2-m] = f[-2]\, g[2-(-2)] + f[-1]\, g[2-(-1)] \\
&= f[-2]\, g[4] + f[-1]\, g[3] = (2)(0)+(2)(1) = 2
\end{aligned}
$$

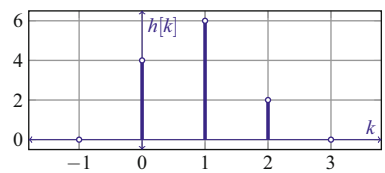
In conclusion, $h[k]=[\underline{4}\,,\,6,\,2]$, see Fig. 4.4. Note that, if given as a list, sample $h[0]$ is underlined.

Method 2: Given graphs in Fig. 4.1, two sequences $f[k]$ and $g[k]$ may be written, for example, as

$$
\begin{aligned}
f[k] &= 2\delta[k+2]+2\delta[k+1] \\
g[k] &= [\underline{\;}\,,\,,\,2,\,1]
\end{aligned}
$$

where the leading empty spaces in $g[k]$ keep the sequence index so that $g[2]=2$ and $g[3]=1$. Underlined space '$\underline{\;}$' is used to indicate the index 0 position. By definition, convolution sum is

Fig. 4.4 Example P.4.3



$$h[k] = f[k] * g[k] = (2\delta[k+2] + 2\delta[k+1]) * g[k] = 2\,g[k+2] + 2\,g[k+1]$$

where, $g[k+2]$ and $g[k+1]$ are shifted $g[k]$ as

$$2\,g[k+2] = 2\,[\underline{2}\,, 1] = [\underline{4}\,, 2]$$
$$2\,g[k+1] = 2\,[\underline{\ }\,, 2, 1] = [\underline{\ }\,, 4, 2]$$

so that $\underline{2}$ syntax is used to mark $k = 0$ index in $g[k+2]$ sequence. The convolution sum $h[k] = 2\,g[k+2] + 2\,g[k+1]$ may be done by the tabular method as

k		-2	-1	0	1	2	3
$2\,g[k+2]$				$\underline{4}$	2		
$2\,g[k+1]$	+				4	2	
$h[k]$				$\underline{4}$	6	2	

which is in agreement with the **Method 1:** result.

4.4. Given sequence $h[n] = [1, \underline{2}\,, 0, -3]$, and Dirac delta function $\delta[n]$, convolution is equivalent to multiplication by one because Dirac delta is the identity function, as

$$y[n] = \delta[n] * h[n] = h[n] * \delta[n] = h[n]$$

4.5. Given $h[n] = [1, \underline{2}\,, 0, -3]$ and sequence of two Dirac delta, as

$$x[n] = \delta[n+1] + \delta[n-2]$$

it follows that

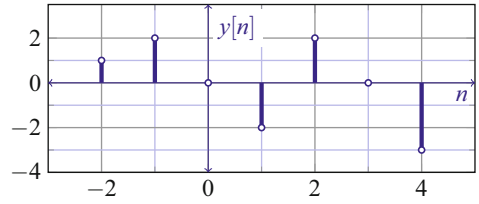
$$\begin{aligned} y[n] = x[n] * h[n] &= (\delta[n+1] + \delta[n-2]) * h[n] \\ &= \delta[n+1] * h[n] + \delta[n-2] * h[n] \\ &= h[n+1] + h[n-2] \end{aligned}$$

Where, two shifted sequences $h[n+1]$ and $h[n-2]$ are

$$h[n] = [1, \underline{2}\,, 0, -3]$$
$$h[n+1] = [1, 2, \underline{0}\,, -3] \quad \text{shifted one places to right}$$
$$h[n-2] = [\underline{\ }\,, 1, 2, 0, -3] \quad \text{shifted two places to left}$$

One way to calculate the convolution sum $y[n] = h[n+1] + h[n-2]$ is by tabular method as

Fig. 4.5 Example P.4.4



n		-2	-1	$\underline{0}$	1	2	3	4
$h[n+1]$		1	2	$\underline{0}$	-3			
$h[n-2]$	$+$				1	2	0	-3
$y[n]$		1	2	$\underline{0}$	-2	2	0	-3

That is to say, (see also Fig. 4.5)

$$y[n] = [1, 2, \underline{0}\ , -2, 2, 0, -3]$$

4.6. Given $h[n] = [1, \underline{2}\ , 0, -3]$ and sequence of three Dirac delta, as

$$x[n] = [\underline{1}\ , 1, 1] = \delta[n] + \delta[n-1] + \delta[n-2]$$

it follows that,

$$\begin{aligned} y[n] = x[n] * h[n] &= (\delta[n] + \delta[n-1] + \delta[n-2]) * h[n] \\ &= h[n] + h[n-1] + h[n-2] \end{aligned}$$

Knowing $h[n] = [1, \underline{2}\ , 0, -3]$ it follows that two shifted sequences are

$$h[n-1] = [\underline{1}\ , 2, 0, -3]$$

$$h[n-2] = [\underline{\ }\ , 1, 2, 0, -3]$$

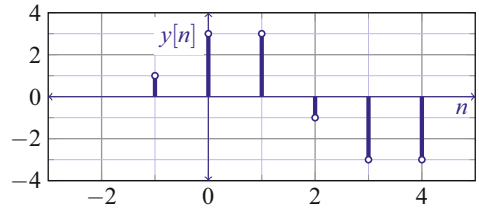
so that the convolution sum $y[n] = h[n] + h[n-1] + h[n-2]$ is

n		-1	$\underline{0}$	1	2	3	4
$h[n]$		1	$\underline{2}$	0	-3		
$h[n-1]$			$\underline{1}$	2	0	-3	
$h[n-2]$	$+$			1	2	0	-3
$y[n]$		1	$\underline{3}$	3	-1	-3	-3

That is to say, (see also Fig. 4.6)

$$y[n] = [1, \underline{3}\ , 3, -1, -3, -3]$$

Fig. 4.6 Example P.4.6



4.7. Given two sequeces

$$x[n] = [2, 1, \underline{-1}, -2, -3]$$
$$h[n] = [1, \underline{2}, 0, -3]$$

the tabular method may be used to calculate convolution

$$y[n] = x[n] * h[n]$$

Noting the index reference for $x[n]$, it may be written as

$$x[n] = 2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]$$

so that

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]) * h[n] \\ &= 2h[n+2] + h[n+1] - h[n] - 2h[n-1] - 3h[n-2] \end{aligned}$$

where the required shifted sequences of $h[n]$ are

$$\begin{aligned} 2\,h[n+2] &= 2\,[1, 2, 0, \underline{-3}\,] = [2, 4, 0, \underline{-6}\,] \\ h[n+1] &= [1, 2, \underline{0}, -3] \\ -h[n] &= [-1, \underline{-2}, 0, 3] \\ -2\,h[n-1] &= -2\,[\underline{1}, 2, 0, -3] = [\underline{-2}, -4, 0, 6] \\ -3\,h[n-2] &= -3\,[\underline{\ }, 1, 2, 0, -3] = [\underline{\ \ \ }, -3, -6, 0, 9] \end{aligned}$$

Above intermediate results are tabulated and added in tabular form as follows

$$y[n] = \sum_{k=-2}^{2} x[k]\,h[n-k]$$

n		-3	-2	-1	0	1	2	3	4
$x[-2]\,h[n+2]$		2	4	0	-6				
$x[-1]\,h[n+1]$			1	2	0	-3			
$x[0]\,h[n]$				-1	-2	0	3		
$x[1]\,h[n-1]$					-2	-4	0	6	
$x[2]\,h[n-2]$	+					-3	-6	0	9
$y[n]$		2	5	1	-10	-10	-3	6	9

Visually, tabular method does the same "sweep" of $h[k-n]$ as already illustrated in the graphics method. In conclusion, (see Fig. 4.7)

$$y[n] = x[n] * h[n] = \sum_{k=-2}^{2} x[k]\,h[n-k] = [2, 5, 1, \underline{-10}, -10, -3, 6, 9]$$

4.8. Given two long sequences $x[k]$ and $h[k]$ in analytical form, that is to say with the infinite non–zero interval, the convolution is resolved as same as in the case of continuous signals. Function transformations may be used to derive $x[m]$ and $h[k-m]$, see Fig. 4.8.

$$x[k] = \alpha^k\,u[k]; \quad (0 < \alpha < 1)$$
$$h[k] = u[k]$$

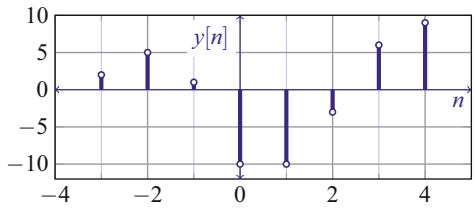
Case 1 : $k < 0$

Given that $k < 0$, all products inside the convolution sum equal zero, see Fig. 4.9, as

$$x[m]\;h[k-m] = 0; \quad \forall m$$

and by consequence, convolution sum equal zero, as

Fig. 4.7 Example P.4.7



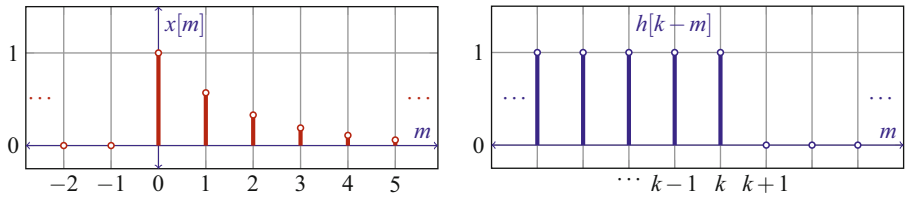


Fig. 4.8 Example P.4.8

Fig. 4.9 Example P.4.8

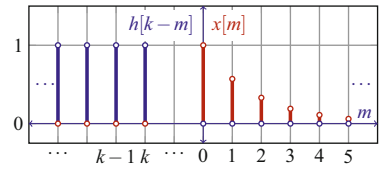
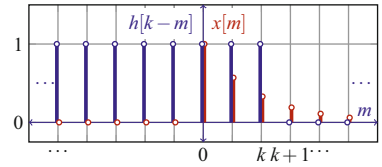


Fig. 4.10 Example P.4.8



$$y[k] = x[k] * h[k] = 0; \quad (k < 0)$$

Case 2 : $k > 0$

Given that $k \geq 0$, there are non–zero products inside the convolution sum interval, see Fig. 4.10, as

$$x[m]\, h[k-m] \neq 0; \quad m = [0, 1, \ldots, k]$$

and by consequence, convolution sum is

$$y[k] = x[k] * h[k] = \sum_{0}^{k} x[m]\, h[k-m]$$

which leads into geometric sum as

$$= \underbrace{\sum_{0}^{k} \alpha^m \; (1)}_{\text{geometric sum}} = \left\{ \sum_{n=0}^{N-1} \alpha^n = \left\{ \begin{array}{ll} N; & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}; & \alpha \neq 1 \end{array} \right\} \right. = \frac{1-\alpha^{k+1}}{1-\alpha} \quad (k \geq 0)$$

Summary: the two cases together may be written in the compact form as

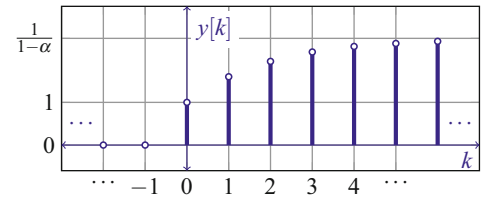
$$y[k] = \frac{1-\alpha^{k+1}}{1-\alpha}\, u[k]$$

Note that due to $\alpha < 1$ then (see Fig. 4.11)

$$\lim_{k\to\infty} \alpha^{k+1} \to 0 \quad \therefore \quad \lim_{k\to\infty} \frac{1-\alpha^{k+1}}{1-\alpha} \to \frac{1}{1-\alpha}$$

4.9. Given port $x[k]$ and exponential $h[k]$ functions in the analytical form, as

Fig. 4.11 Example P.4.8



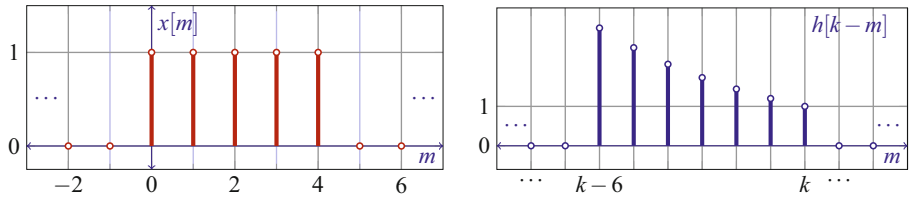


Fig. 4.12 Example P.4.9

$$x[k] = \begin{cases} 1; & (0 \le k \le 4) \\ 0; & \text{otherwise} \end{cases} \qquad \text{and} \qquad h[k] = \begin{cases} \alpha^k; & (0 \le k \le 6) \\ 0; & \text{otherwise} \end{cases}$$

by definition, convolution sum is $y[k] = x[k] * h[k] = \sum_{m=-\infty}^{\infty} x[m]\, h[k-m]$. After function transformations $x[m]$ and $h[k-m]$ are as in Fig. 4.12.

Case 1 : $k < 0$

For $k < 0$ all products inside the convolution sum equal zero, see Fig. 4.13, as

$$x[m]\, h[k-m] = 0; \quad \forall m$$

and by consequence, convolution sum equal zero, as

$$y[k] = x[k] * h[k] = 0; \quad (k < 0)$$

Case 2 : $0 \le k \le 4$

As the leading sample h_k of $h[k-m]$ enters interval $[0, 4]$ defined by port function $x[m]$, non–zero products are in $[0, k]$ interval, see Fig. 4.14, thus convolution sum is

$$y[k] = x[k] * h[k] = \sum_{m=0}^{k} x[m]\, h[k-m] = \sum_{m=0}^{k} (1)\, \alpha^{k-m} = \sum_{m=0}^{k} \alpha^{k-m}$$

$$= \left\{ \begin{array}{lll} r = k - m & \Rightarrow & m = 0 \;\; \therefore \;\; r = k \\ & & m = k \;\; \therefore \;\; r = 0 \end{array} \right\} = \sum_{r=k}^{0} \alpha^r = \sum_{r=0}^{k} \alpha^r = \frac{1 - \alpha^{k+1}}{1 - \alpha}$$

Fig. 4.13 Example P.4.9

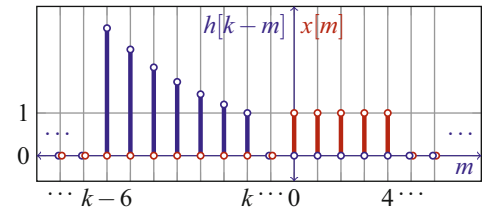


Fig. 4.14 Example P.4.9

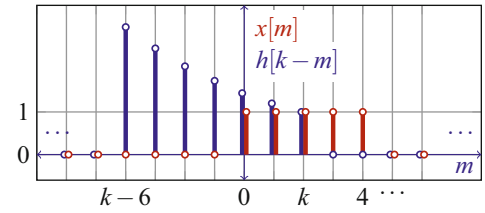
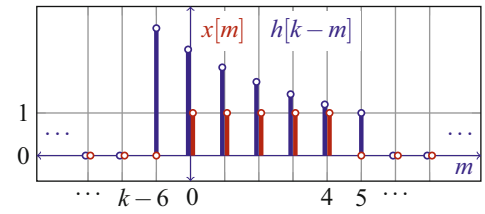


Fig. 4.15 Example P.4.9



Case 3 : $5 \le k \le 6$

As the leading sample h_k of $h[k-m]$ leaves interval $[0, 4]$ defined by port function $x[m]$, but h_{k-4} is inferior to zero, (note that exponential suite has six and port has four samples) i.e.

$$\left.\begin{array}{rll} k & > 4 & \\ k-6 & \le 0 \quad \therefore & k \le 6 \end{array}\right\} \Rightarrow k = (5, 6)$$

Therefore, non–zero products are in interval $[0, 4]$, see Fig. 4.15, and convolution sum is

$$y[k] = x[k] * h[k] = \sum_{m=0}^{4} x[m]\, h[k-m] = \sum_{m=0}^{4} (1)\, \alpha^{k-m} = \sum_{m=0}^{4} \alpha^k\, \alpha^{-m}$$

$$= \alpha^k \sum_{m=0}^{4} \left(\alpha^{-1}\right)^m = \alpha^k \frac{1 - \left(\alpha^{-1}\right)^5}{1 - \alpha^{-1}} = \frac{\alpha^k - \alpha^{k-5}}{1 - \alpha^{-1}} \quad \frac{\alpha}{\alpha} = \frac{\alpha^{k+1} - \alpha^{k-4}}{\alpha - 1}$$

Case 4 : $7 \le k \le 10$

As the trailing sample h_k of $h[k-m]$ enters interval $[0, 4]$ defined by port function $x[m]$,

Fig. 4.16 Example P.4.9

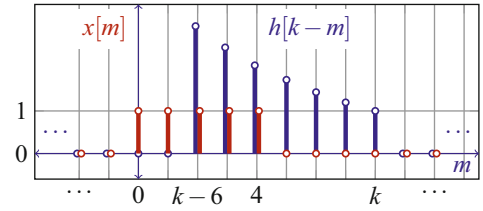
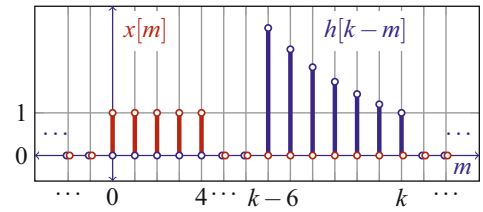


Fig. 4.17 Example P.4.9



$$\left.\begin{array}{lll} k-6 & > 0 & \therefore \quad k > 6 \\ k-6 & \leq 4 & \therefore \quad k \leq 10 \end{array}\right\} \Rightarrow k = (7, 8, 9, 10)$$

and non–zero products are in interval $[k-6, 4]$, see Fig. 4.16, thus convolution sum is

$$y[k] = x[k] * h[k] = \sum_{m=k-6}^{4} x[m]\, h[k-m] = \sum_{m=k-6}^{4} (1)\, \alpha^{k-m}$$

$$= \left\{\begin{array}{ll} r = m-k+6; & k-m = 6-r \\ & m = k-6 \quad \therefore \quad r = 0 \\ & m = 4 \quad \therefore \quad r = 10-k \end{array}\right\}$$

$$= \sum_{r=0}^{10-k} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-k} \left(\alpha^{-1}\right)^r = \alpha^6 \frac{1-\alpha^{k-11}}{1-\alpha^{-1}} = \frac{\alpha^7 - \alpha^{k-4}}{\alpha - 1}$$

Case 5 : $k > 10$

As the trailing sample h_{k-6} of $h[k-m]$ leaves interval $[0, 4]$ defined by port function $x[m]$, that is $k-6 > 4 \quad \therefore \quad k > 10$, see Fig. 4.17, thus all products equal zero, as

$$x[m]\, h[k-m] = 0; \quad \forall m$$

and by consequence, convolution sum equal zero, as

$$y[k] = x[k] * h[k] = 0; \quad (k > 10)$$

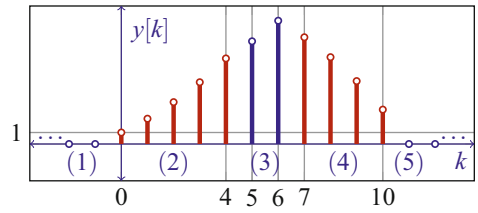


Fig. 4.18 Example P.4.9

Summary: see Fig. 4.18

$$y[k] = \begin{cases} 0\,; & k < 0 \\ \dfrac{1-\alpha^{k+1}}{1-\alpha}\,; & 0 \le k \le 4 \\ \dfrac{\alpha^{k+1}-\alpha^{k-4}}{\alpha-1}\,; & 5 \le k \le 6 \\ \dfrac{\alpha^{7}-\alpha^{k-4}}{\alpha-1}\,; & 7 \le k \le 10 \\ 0\,; & k > 10 \end{cases}$$

4.10. Given two sequences $x[n]$ and $y[n]$ in analytical form and assuming finite n, see Fig. 4.19, convolution

$$h[n] = x[n] * y[n]$$

may be resolved analytically as follows.

For $n < 0$ all convolution products equal zero, thus (assuming finite m) for $n \ge 0$

$$h[n] = x[n] * y[n] = \sum_{-m}^{m} x[m]\; y[n-m] = 0 + \sum_{0}^{m} \sin(\omega_0 m)\;(1) = \sum_{0}^{m} \sin(\omega_0 m)$$

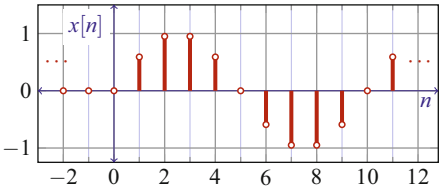
$$= \left\{ \sin(jx) = \frac{e^{jx}-e^{-jx}}{2j},\quad \cos(jx) = \frac{e^{jx}+e^{-jx}}{2} \right\} = \frac{1}{2j}\sum_{0}^{m} e^{j\,\omega_0 m} - \frac{1}{2j}\sum_{0}^{m} e^{-j\,\omega_0 m}$$

$$= \left\{ e^{ab} = (e^a)^b,\quad e^{a+b} = e^a\, e^b \right\} = \frac{1}{2j}\sum_{0}^{m} \left(e^{j\,\omega_0}\right)^m - \frac{1}{2j}\sum_{0}^{m} \left(e^{-j\,\omega_0}\right)^m$$

$$= \left\{ \sum_{0}^{m} a^n = \frac{1-a^{m+1}}{1-a} \right\} = \frac{1}{2j}\,\frac{1-e^{j\,\omega_0\,(m+1)}}{1-e^{j\,\omega_0}} - \frac{1}{2j}\,\frac{1-e^{-j\,\omega_0\,(m+1)}}{1-e^{-j\,\omega_0}}$$

$$= \left\{ 1 = e^0 = e^{j\,\omega_0\left(\frac{m+1}{2}\right) - j\,\omega_0\left(\frac{m+1}{2}\right)} = e^{j\,\omega_0\left(\frac{m+1}{2}\right)}\; e^{-j\,\omega_0\left(\frac{m+1}{2}\right)} \right\}$$

$$= \frac{1}{2j}\,\frac{e^{j\,\omega_0\left(\frac{m+1}{2}\right)}\; e^{-j\,\omega_0\left(\frac{m+1}{2}\right)} - e^{j\,\omega_0\,(m+1)}}{e^{\frac{j\,\omega_0}{2}}\; e^{\frac{-j\,\omega_0}{2}} - e^{j\,\omega_0}} - \frac{1}{2j}\,\frac{e^{j\,\omega_0\left(\frac{m+1}{2}\right)}\; e^{-j\,\omega_0\left(\frac{m+1}{2}\right)} - e^{-j\,\omega_0\,(m+1)}}{e^{\frac{j\,\omega_0}{2}}\; e^{\frac{-j\,\omega_0}{2}} - e^{-j\,\omega_0}}$$



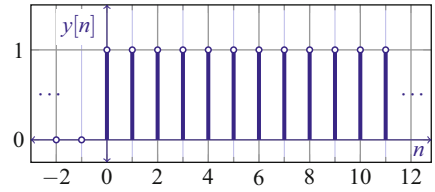
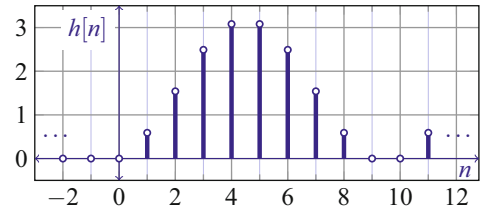


Fig. 4.19 Example P.4.10

Fig. 4.20 Example P.4.10



$$= \frac{1}{2j}\,\frac{e^{j\,\omega_0\left(\frac{m+1}{2}\right)}\left(e^{-j\,\omega_0\left(\frac{m+1}{2}\right)} - e^{j\,\omega_0\left(\frac{m+1}{2}\right)}\right)}{e^{\frac{j\,\omega_0}{2}}\left(e^{-\frac{j\,\omega_0}{2}} - e^{\frac{j\,\omega_0}{2}}\right)} - \frac{1}{2j}\,\frac{e^{-j\,\omega_0\left(\frac{m+1}{2}\right)}\left(e^{j\,\omega_0\left(\frac{m+1}{2}\right)} - e^{-j\,\omega_0\left(\frac{m+1}{2}\right)}\right)}{e^{-\frac{j\,\omega_0}{2}}\left(e^{\frac{j\,\omega_0}{2}} - e^{-\frac{j\,\omega_0}{2}}\right)}$$

$$= \frac{1}{2j}\,\frac{e^{\frac{j\,\omega_0\,m}{2}}\cancel{e^{\frac{j\,\omega_0}{2}}}}{\cancel{e^{\frac{j\,\omega_0}{2}}}}\,\frac{\cancel{-2j}\,\sin\frac{\omega_0(m+1)}{2}}{\cancel{-2j}\,\sin\frac{\omega_0}{2}} - \frac{1}{2j}\,\frac{e^{-\frac{j\,\omega_0\,m}{2}}\cancel{e^{-\frac{j\,\omega_0}{2}}}}{\cancel{e^{-\frac{j\,\omega_0}{2}}}}\,\frac{\cancel{2j}\,\sin\frac{\omega_0(m+1)}{2}}{\cancel{2j}\,\sin\frac{\omega_0}{2}}$$

$$= \frac{\sin\frac{\omega_0(m+1)}{2}}{\sin\frac{\omega_0}{2}}\,\frac{1}{2j}\left(e^{\frac{j\,\omega_0\,m}{2}} - e^{-\frac{j\,\omega_0\,m}{2}}\right) = \frac{\sin\frac{\omega_0(m+1)}{2}}{\sin\frac{\omega_0}{2}}\,\frac{1}{\cancel{2j}}\,\cancel{2j}\,\sin\frac{\omega_0\,m}{2}$$

$$= \frac{\sin\frac{\omega_0(m+1)}{2}}{\sin\frac{\omega_0}{2}}\,\sin\frac{\omega_0\,m}{2};\quad (0 \le m \le n)$$

The convolution sequence $h[n]$ is shown in Fig. 4.20

4.2 Discrete Signal Energy and Power

4.11. Given Dirac function, $x[k] = \delta[k]$, by definition

$$E[k] = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} |\delta[k]|^2 = \cdots + 0 + 0 + \cdots + 1^2 + 0 + 0 + \cdots$$
$$= 1 < \infty$$

which is to say that since the energy is distributed over infinitely long time interval $k = \infty$, therefore at any given moment the available average power must be

$$\langle P[k]\rangle = \lim_{k\to\infty}\frac{E[k]}{k} = \lim_{k\to\infty}\frac{1}{k} = 0$$

as same as for continuous functions where $t \to \infty$.

4.12. Given infinitely long series $x[k] = \left(\frac{1}{4}\right)^k u[k]$, it follows that

$$E[k] = \sum_{k=-\infty}^{\infty}|x[k]|^2 = \sum_{k=-\infty}^{\infty}\left|\left(\frac{1}{4}\right)^k u[k]\right|^2 = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{2k} = \sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k$$

$$\left\{\text{geometric series,}\quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}\quad \text{where,}\quad |r|<1\right\}$$

$$\left\{\text{as well as,}\quad \sum_{k=0}^{\infty} ar^{2k} = \frac{a}{1-r^2}\right\}$$

$$\left\{\text{or,}\quad \sum_{k=0}^{\infty} ar^{2k+1} = \frac{a\,r}{1-r^2}\right\}$$

$$\left\{\therefore\ a=1\quad\text{and}\quad r=\frac{1}{16}\right\}$$

$$= \frac{1}{1-\frac{1}{16}} = \frac{16}{15} < \infty$$

which is to say that since the energy is distributed over infinitely long time interval $k = \infty$, therefore

$$\langle P[k]\rangle = \lim_{k\to\infty}\frac{E[k]}{k} = \lim_{k\to\infty}\frac{16/15}{k} = 0$$

as same as for continuous functions where $t \to \infty$.

4.13. Samples of complex exponential series, $x[k] = e^{j\,10k}\,u[k]$, are added for $k \in [-n, n]$, consequently there are in total $2n+1$ samples to add. That is, $k = 1, 2, 3, \ldots, n$ (therefore n points in total) on the positive side, plus $k = -1, -2, -3, \ldots, -n$ (i.e. n points in total) on the negative side, plus $k = 0$ (i.e. one additional point) in the middle, thus $2n+1$ points in total. Accordingly, by definition for power we find

$$\langle P[k]\rangle = \lim_{n\to\infty}\left[\frac{1}{2n+1}\sum_{k=-n}^{n}|x[k]|^2\right] = \lim_{n\to\infty}\left[\frac{1}{2n+1}\sum_{k=-n}^{n}\left|e^{j\,10k}\,u[k]\right|^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2n+1}\sum_{k=0}^{n}\left|e^{j\,10k}\right|^2\right] = \lim_{n\to\infty}\left[\frac{1}{2n+1}\underbrace{\sum_{k=0}^{n}|1_k|^2}_{n+1}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n+1}{2n+1}\right] = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2} < \infty$$

which is to say that in order to support non–zero average power over infinite long interval $k = \infty$, there must be $E[k] = \langle P[k] \rangle \, k = \infty$.

4.14. Infinite sum of power series where $|\alpha| < 1$ is convergent, thus by definition

$$E[k] = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} \left|\alpha^k \, u[k]\right|^2 = \sum_{k=0}^{\infty} \left|\alpha^k\right|^2 = \underbrace{\sum_{k=0}^{\infty} \left(\alpha^2\right)^k}_{\text{geometric series}}$$

$$= \frac{1}{1-\alpha^2} < \infty$$

Finite amount of energy must be distributed over infinitely long interval, thus the average power equal

$$\langle P[k] \rangle = \lim_{k \to \infty} \frac{E[k]}{k} = \lim_{k \to \infty} \frac{\dfrac{1}{1-\alpha^2}}{k} = 0$$

4.15. Infinitely long step function has average power calculated as

$$\langle P[k] \rangle = \lim_{n \to \infty} \left[\frac{1}{2n+1} \sum_{k=-n}^{n} |x[k]|^2 \right] = \lim_{n \to \infty} \left[\frac{1}{2n+1} \sum_{k=-n}^{n} |u[k]|^2 \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{2n+1} \underbrace{\sum_{k=0}^{n} |1_k|^2}_{n+1} \right] = \lim_{n \to \infty} \left[\frac{n+1}{2n+1} \right] = \left(\frac{\infty}{\infty} \right) = LH \lim_{n \to \infty} \frac{1}{2}$$

$$= \frac{1}{2} < \infty$$

which is to say that in order to support non–zero average power over infinite long interval $k = \infty$, there must be $E[k] = \langle P[k] \rangle \, k = \infty$.

4.16. Average power of an infinitely long periodic series is calculated over one period as

$$\langle P[k] \rangle = \lim_{n \to \infty} \left[\frac{1}{2n+1} \sum_{k=-n}^{n} |x[k]|^2 \right] = \lim_{n \to \infty} \left[\frac{1}{2n+1} \sum_{k=-n}^{n} \left| \cos\left[\frac{\pi}{6} k \right] \right|^2 \right]$$

$$= \left\{ \cos^2(x) = \left(\frac{1 + \cos(2x)}{2} \right) \right\}$$

$$= \lim_{n \to \infty} \left[\frac{1}{2n+1} \sum_{k=-n}^{n} \left[\frac{1}{2} + \frac{1}{2} \cos\left[2 \frac{\pi}{6} k \right] \right] \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{2n+1} \, \frac{1}{2} \underbrace{\sum_{k=-n}^{n} 1_k}_{2n+1} + \frac{1}{2n+1} \, \frac{1}{2} \underbrace{\sum_{k=-n}^{n} \cos\left[\frac{\pi}{3} k \right]}_{=0} \right]$$

$$= \left\{ \begin{array}{l} \text{in total, there are (2n+1) terms of 1/2 each;} \\ \text{the sum of sin or cos over one period equal zero ;} \end{array} \right\}$$

$$= \lim_{n\to\infty} \left[\frac{1}{\cancel{2n+1}} \; \frac{1}{2} \; \cancel{(2n+1)} \right] = \frac{1}{2} < \infty$$

which is to say that in order to support non–zero average power over infinite long interval $k = \infty$, there must be $E[k] = \langle P[k] \rangle \, k = \infty$.

Continuous Time Fourier Transform

5

Classification of Fourier transformations, sums and series:

Continuous time (CT) periodic, Fig. 5.1 (left), and nonperiodic, Fig. 5.1 (right), $x(t)$ functions. Periodic functions are transformed by continuous-time Fourier series (CT FS), while continuous time nonperiodic functions are transformed by continuous-time Fourier transformation (CT FT).
Similarly, discrete-time (DT) periodic functions are mapped by discrete-time Fourier series (DT FS), while discrete-time nonperiodic functions are transformed by discrete-time Fourier transformation (DT FS).

In addition, in order to carry out Fourier transformation (FT) of continuous signals on numerical processors, CT signals must be sampled before applying discrete Fourier transformation (DFT) algorithm. Version of DFT algorithm that is optimized for fast execution is referred to as fast Fourier transformation (FFT).

Fourier and inverse Fourier integrals are defined as

$$\mathscr{F}\big(x(t)\big) \stackrel{\text{def}}{=} X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\, t}\, dt \tag{5.1}$$

$$\mathscr{F}^{-1}\big(X(\omega)\big) \stackrel{\text{def}}{=} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\,\omega\, t} d\omega \tag{5.2}$$

Fourier series may be defined in various forms. The most common form is a periodic function $f(x)$ development, where the resulting series is in the form of an infinite sum of sines and cosines terms, as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \tag{5.3}$$

where coefficients a_n and b_n are defined as

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_5

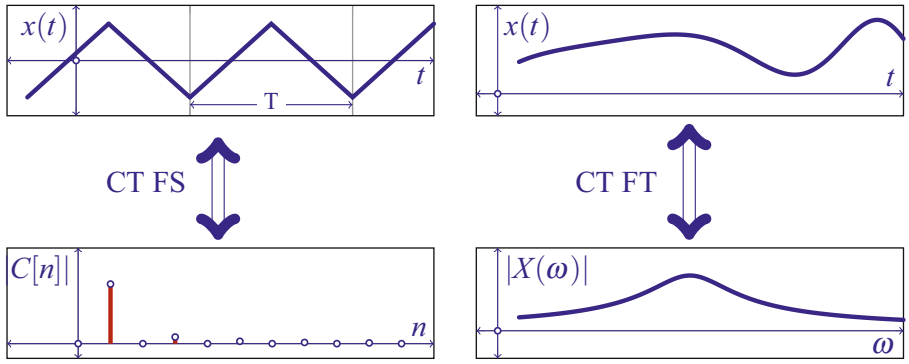


Fig. 5.1 Comparison between CT FS and CT FT

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \tag{5.4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \quad (n \geq 1) \tag{5.5}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \quad (n \geq 1) \tag{5.6}$$

Note: the resulting infinite series may or may not be convergent.
Properties of Fourier Transformation

1. *Linearity:* given three functions and their FT as $f(t) \overset{\text{FT}}{\rightarrow} F(\omega)$, $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$, and $y(t) \overset{\text{FT}}{\rightarrow} Y(\omega)$, and constants a, b, then

$$f(t) = ax(t) + b(y(t) \quad \Rightarrow \quad F(\omega) = aX(\omega) + bY(\omega)$$

2. *Time shifting:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$ and a constant τ, then

$$x(t-\tau) \overset{\text{FT}}{\rightarrow} e^{-j\tau\omega}\,X(\omega)$$

That is to say, time delay $(t-\tau)$ translates into phase shift $e^{-j\tau\omega}$ in frequency domain.

3. *Frequency shifting:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$ and a constant ω_0, then

$$e^{-j\,\omega_0\,t}\,x(t) \overset{\text{FT}}{\rightarrow} X(\omega-\omega_0)$$

That is to say, phase shift $e^{-j\,\omega_0\,t}$ in time domain translates into frequency shift $(\omega-\omega_0)$.

4. *Time scaling:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$ and a constant a, then

$$x(at) \overset{\text{FT}}{\rightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

5. *Derivative property:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$, then FT of (n)–th derivative is

$$x^{(n)}(t) \overset{\text{FT}}{\rightarrow} (-j\omega)^n X(\omega)$$

6. *Multiplication by t^n:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$, then

$$t^n x(at) \overset{\text{FT}}{\rightarrow} (-j)^n \frac{d^n}{d\omega^n} X(\omega)$$

7. *Convolution property:* given $x(t) \overset{\text{FT}}{\rightarrow} X(\omega)$ and $y(t) \overset{\text{FT}}{\rightarrow} Y(\omega)$, then

$$x(t) * y(t) \overset{\text{FT}}{\rightarrow} X(\omega)\, Y(\omega)$$

Problems

5.1 Special Functions

Given analytical functions $x(t)$ in P.5.1 to P.5.15 (where applies, assume $a, \tau > 0$), derive Fourier transformations $X(\omega)$.

5.1. $x(t) = \delta(t)$

5.2. $x(t) = \delta(t - \tau)$

5.3. $x(t) = a$

5.4. $x(t) = e^{j\,\omega_0\, t}$

5.5. $x(t) = \cos(\omega_0\, t)$

5.6. $x(t) = \sin(\omega_0\, t)$

5.7. $x(t) = e^{-at}\, u(t)$

5.8. $x(t) = e^{at}\, u(-t)$

5.9. $x(t) = e^{-a|t|}$

5.10. $x(t) = a\, \Pi\left(\frac{t}{\tau}\right)$

5.11. $x(t) = a\, \Lambda\left(\frac{t}{2\tau}\right)$

5.12. $x(t) = \text{sign}\,(t)$

5.13. $x(t) = u(t)$

5.14. $x(t) = e^{-t^2}$

5.15. $x(t) = \text{sinc}\,(t)$

5.2 Composite Functions

Given graphs of composite functions in P.5.16 to P.5.21, derive FT.

5.16. Given graph of function $x(t)$ in Fig. 5.2, derive $X(\omega)$.

5.17. Given graph of function $x(t)$ in Fig. 5.3 derive $X(\omega)$.

5.18. Given graph of RF pulse $h(t)$ where $\omega_0 = 2\pi f_0$ is the RF frequency, a is sine amplitude, and τ is duration of RF pulse, see Fig. 5.4, derive its Fourier transform.

Fig. 5.2 Example P.5.16

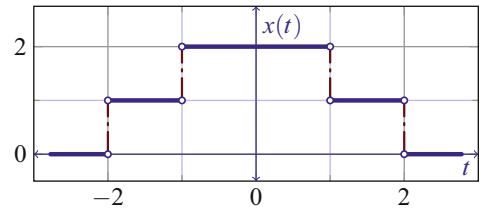


Fig. 5.3 Example P.5.17

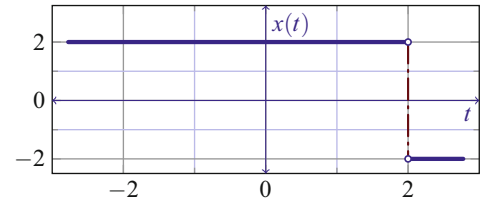


Fig. 5.4 Example P.5.18

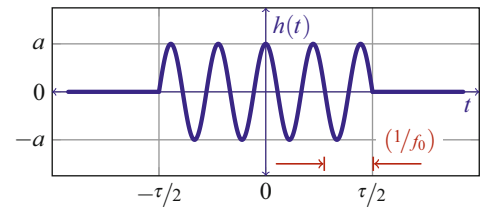
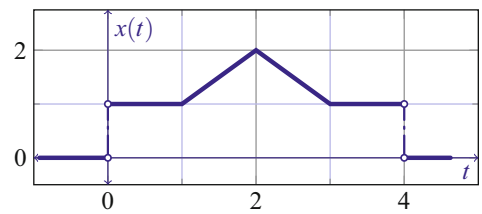


Fig. 5.5 Example P.5.19



5.19. Given graph of $x(t)$ in Fig. 5.5, derive its Fourier transform.

5.20. Given graph of $x(t)$ in Fig. 5.6, derive its Fourier transform.

5.21. Given graph of $x(t)$ in Fig. 5.7, derive its Fourier transform.

5.22. Given graph of $x(t)$ in Fig. 5.8, derive its Fourier transform.

5.3 Fourier Series

Unless specified otherwise, derive Fourier series within $(-\pi \leq t \leq \pi)$ of functions $x(t)$ in P.5.23 to P.5.33.

5.23. a

5.24. $\frac{a}{\pi} t$

5.25. $|\sin t|$

Fig. 5.6 Example P.5.20

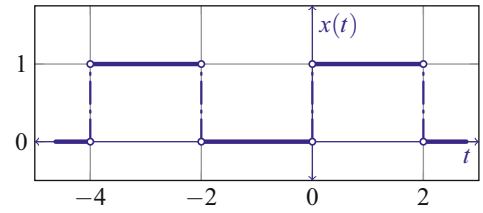


Fig. 5.7 Example P.5.21

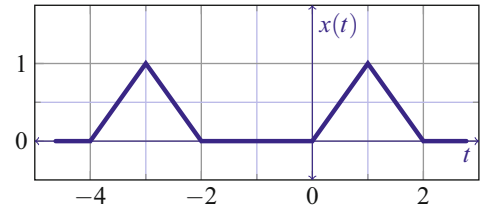
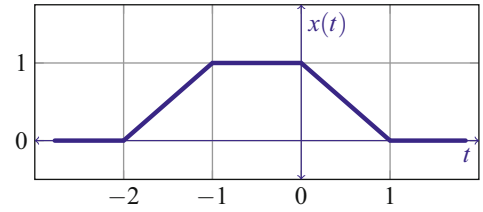


Fig. 5.8 Example P.5.22



5.26. $\sin^3 t$

5.27. e^t

5.28. $a\,\Pi\left({}^{(t-\pi/2)}/_{\pi}\right), x \in (0, 2\pi)$

5.29. $a\,\Pi\left({}^{t}/_{\pi}\right)$

5.30. t^2/pi^2

5.31. $\pi^2/16\left(\pi^2 - t^2\right)$

5.32. $-\dfrac{2a}{\pi}\,|t| + a$

5.33. $\dfrac{a}{\pi}\,t,\ (0 \le t \le \pi)$

$-\dfrac{a}{\pi}\,(t-\pi) + a,\ (\pi \le t \le 2\pi)$

Answers

5.1 Special Functions

5.1. Dirac functions equal zero for all arguments t, except in $t = 0$ point. Its Fourier transformation is derived by definition (5.1) as

$$X(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\, t}\, dt = \left\{\delta(t) \neq 0 \Rightarrow t = 0 \quad \therefore \quad e^{j\omega\, 0} = 1\right\}$$
$$= \int_{-\infty}^{\infty} \delta(t) \cancel{e^{-j\,\omega\, t}}^{\,1}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt \stackrel{\text{def}}{=} 1$$

These two functions, Dirac delta and a constant, Fig. 5.9, are said to be "dual" because Fourier transform of Dirac function is a constant function, and vice versa, i.e. Fourier transform of a constant function is Dirac function. Physical interpretation is that in order to synthesize Dirac function it is necessary to add infinitely $\sin(\omega\, t)$, and/or $\cos(\omega\, t)$, functions of equal powers whose frequencies are in the range $\omega \in [-\infty, \infty]$.

5.2. Fourier transformation of a delayed Dirac function $\delta(t - \tau)$ is derived by definition as

$$X(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\, t}\, dt = \left\{\delta(t - \tau) \neq 0 \Rightarrow t = \tau\right\}$$
$$= \int_{-\infty}^{\infty} \delta(t - \tau)\, e^{-j\omega\,\tau}\, dt = e^{-j\omega\,\tau} \underbrace{\int_{-\infty}^{\infty} \delta(t - \tau)\, dt}_{\stackrel{\text{def}}{=} 1}$$
$$= (1)\, e^{-j\omega\,\tau}$$

That is, see A.5.1, delaying function $x(t)$ by τ in time domain is equivalent to multiplying Fourier transformation $X(\omega)$ by $\exp(-j\omega\,\tau)$ in frequency domain (i.e. it adds phase term).

Reminder: Fourier transformation of a delayed function $f(t - \tau)$ is equivalent to Fourier transformation of its non–delayed version $\mathscr{F}\big(f(t)\big)$ multiplied by the phase delayed term, as

$$\mathscr{F}\big(f(t - \tau)\big) = \mathscr{F}\big(f(t)\big)\, e^{-j\omega\,\tau}$$

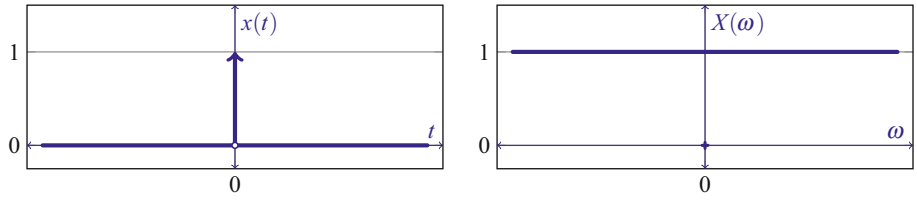


Fig. 5.9 Example P.5.1

5.3. Fourier transformation of a constant function may be derived, for example, by using "reverse engineering" technique, i.e by starting with the inverse Fourier transformation integral (5.2)

$$\mathscr{F}^{-1}(X(\omega)) = x(t) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\,\omega\, t}\, d\omega$$

Given that $x(t) = a$, its transformation function $X(\omega)$ is therefore found as

$$a \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X(\omega)}\ e^{j\,\omega\, t}\, d\omega \tag{5.7}$$

The idea is to gradually deduce the form of $X(\omega)$ until the inverse Fourier equality (5.7) is satisfied, i.e. its right side of the equation is reduced to a. Evidently, it must be that $X(\omega) = 2\pi \times ?$ so that the already existing $1/2\pi$ term may be canceled, as

$$a = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \underline{\cancel{2\pi}\times ?}\ \ e^{j\,\omega\, t}\, d\omega$$

Also, it must be that $X(\omega) = 2\pi\, a \times ?$, i.e.

$$a = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{2\pi\, a \times ?}\ \ e^{j\,\omega\, t}\, d\omega$$

$$\therefore$$

$$a = \frac{1}{2\pi}\, 2\pi\, a \underbrace{\int_{-\infty}^{\infty} \underline{\ ?\ }\ \ e^{j\,\omega\, t}\, d\omega}_{=1} \tag{5.8}$$

Note that after 2π terms cancel and (5.8) is reduced to $a = a\times$ integral, it is possible only if the integral equal one. As already seen in A.5.1 the exponential term is cancelled by exploiting the sampling property of Dirac function because $\delta(\omega) = 0$ for all $\omega \neq 0$, so that the product

$$\delta(\omega)\, e^{-j\,\omega\, t} = \begin{cases} 0 & \forall \omega \neq 0 \\ \delta(\omega)\, e^{0} & \omega = 0 \end{cases}$$

and therefore, by definition

$$\int_{-\infty}^{\infty} \delta(\omega)\, e^{-j\,\omega\, t}\, d\omega = \int_{-\infty}^{\infty} \delta(\omega)\, \cancel{e^{0}}^{\,1}\, d\omega \stackrel{\text{def}}{=} 1$$

That is to say, it has to be that

$$a = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, a\ \underline{\delta(\omega)}\ \ e^{j\,\omega\, t}\, d\omega$$

In conclusion,

$$a \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{X(\omega)}\ e^{j\,\omega\,t}\ d\omega$$

$$\therefore$$

$$a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{2\pi a\ \delta(\omega)}\ e^{j\,\omega\,t}\ d\omega$$

$$\therefore$$

$$\underline{X(\omega) = 2\pi a\ \delta(\omega)}$$

Thus, Fourier transformation of a constant 'a' is indeed proportional to Dirac function (the $2\pi a$ term is the multiplying constant), as derived in A.5.1 and illustrated in Fig. 5.9.

5.4. Amplitude of complex exponential function $x(t) = e^{j\,\omega_0\,t}$ is constant (i.e. equal one, as the radius of a unit circle). Note that ω_0 is a constant as oppose to ω that is variable, thus (see A.5.3)

$$\mathscr{F}^{-1}(X(\omega)) = x(t) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\,\omega\,t}\ d\omega$$

$$\therefore$$

$$e^{j\,\omega_0\,t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{X(\omega)}\ \ e^{j\,\omega\,t}\ d\omega$$

$$\therefore$$

$$e^{j\,\omega_0\,t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{2\pi \times ?}\ \ e^{j\,\omega\,t}\ d\omega$$

In order to factor out $\exp(j\,\omega_0\,t)$ from the right side integral, it is necessary to sample $\exp(j\,\omega\,t)$ at $\omega = \omega_0$ (note that $\exp(j\,\omega_0\,t)$ is *not* function of ω). That is equivalent of multiplying $\exp(j\,\omega\,t)$ by delayed Dirac function $\delta(\omega - \omega_0)$, as

$$e^{j\,\omega_0\,t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{2\pi\ \times \delta(\omega - \omega_0)}\ e^{j\,\omega\,t}\ d\omega$$

$$\therefore$$

$$e^{j\,\omega_0\,t} = \frac{1}{\cancel{2\pi}}\,\cancel{2\pi} \int_{-\infty}^{\infty} \underline{\delta(\omega - \omega_0)}\ e^{j\,\omega_0\,t}\ d\omega$$

$$\therefore$$

$$e^{j\,\omega_0\,t} = e^{j\,\omega_0\,t} \underbrace{\int_{-\infty}^{\infty} \underline{\delta(\omega - \omega_0)}\ d\omega}_{\overset{\text{def}}{=} 1}$$

$$\therefore$$

$$e^{j\,\omega_0\,t} = e^{j\,\omega_0\,t}\ \checkmark$$

In conclusion, given $x(t) = e^{j\,\omega_0\,t}$,

$$X(\omega) = \mathscr{F}(x(t)) = 2\pi\ \delta(\omega - \omega_0)$$

5.5. Application of Euler's formula transforms cosine functions into the sum of complex exponential functions, as

$$x(t) = \cos(\omega_0\, t) = \frac{e^{j\,\omega_0\, t} + e^{-j\,\omega_0\, t}}{2}$$

Therefore,

$$X(\omega) = \mathscr{F}(x(t)) = \mathscr{F}\left[\frac{e^{j\,\omega_0\, t} + e^{-j\,\omega_0\, t}}{2}\right] = \mathscr{F}\left(\frac{e^{j\,\omega_0\, t}}{2}\right) + \mathscr{F}\left(\frac{e^{-j\,\omega_0\, t}}{2}\right)$$

Fourier transformation of a complex exponential function is already solved in A.5.4, thus

$$\mathscr{F}\left(\frac{e^{j\,\omega_0\, t}}{2}\right) = 2\pi\ \delta(\omega - \omega_0)$$

$$\therefore$$

$$\mathscr{F}\left(\frac{e^{-j\,\omega_0\, t}}{2}\right) = 2\pi\ \delta(\omega + \omega_0)$$

In conclusion,

$$X(\omega) = \frac{1}{2}\left[2\pi\ \delta(\omega - \omega_0) + 2\pi\ \delta(\omega + \omega_0)\right] = \pi\ \left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$$

That is to say, Fourier transform $X(\omega)$ of $x(t) = \cos(\omega_0\, t)$ function is the sum of two Dirac functions, one delayed and the other advanced by ω_0, Fig. 5.10. Note that, $X(\omega) \in \mathbb{R}$. This symmetrical form is known as "bilateral", where ω_0 argument is mathematically abstract therefore can be both positive and negative. In engineering applications, specifically if ω_0 argument is used to represent physical frequency (that can not be negative), bilateral form is transformed into its unilateral equivalent so that $\omega_0 \geq 0$.

5.6. Application of Euler formula transforms sine function to the difference of complex exponential functions.

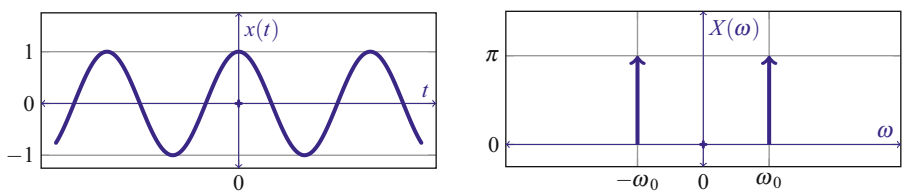


Fig. 5.10 Example P.5.5

$$x(t) = \sin(\omega_0\, t) = \frac{e^{j\,\omega_0\, t} - e^{-j\,\omega_0\, t}}{2j}$$

Therefore,

$$X(\omega) = \mathscr{F}(x(t)) = \mathscr{F}\left[\frac{e^{j\,\omega_0\, t} - e^{-j\,\omega_0\, t}}{2j}\right] = \mathscr{F}\left(\frac{e^{j\,\omega_0\, t}}{2j}\right) - \mathscr{F}\left(\frac{e^{-j\,\omega_0\, t}}{2j}\right)$$

In A.5.4 Fourier transformation of a complex exponential function is already derived, thus

$$\mathscr{F}\left(\frac{e^{j\,\omega_0\, t}}{2}\right) = 2\pi\ \delta(\omega - \omega_0)$$

$$\therefore$$

$$\mathscr{F}\left(\frac{e^{-j\,\omega_0\, t}}{2}\right) = 2\pi\ \delta(\omega + \omega_0)$$

In conclusion,

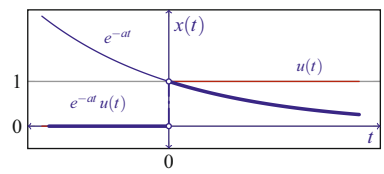
$$\begin{aligned} X(\omega) &= \frac{1}{\cancel{2}j}\left[\cancel{2}\pi\ \delta(\omega - \omega_0) - \cancel{2}\pi\ \delta(\omega + \omega_0)\right] \\ &= \frac{\pi}{j}\ \frac{j}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] \\ &= j\pi\ \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right] \end{aligned}$$

That is to say, Fourier transform of $\sin(\omega_0\ t)$ function is also the sum of two Dirac functions, one delayed and the other advanced by ω_0. However, there is also phase rotation by $\pi/2$ (which is due to j term) as well as the negative $\delta(\omega - \omega_0)$ term. Note that, $X(\omega) \in \mathbb{C}$.

5.7. Sketch of exponential e^{-at} and $u(t)$ functions product is in Fig. 5.11. By definition, Fourier integral of $x(t)$, $a > 0$, is

$$\begin{aligned} X(\omega) &\overset{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\ e^{-j\,\omega\, t}\ dt \\ &= \int_{-\infty}^{\infty} e^{-at} u(t)\ e^{-j\,\omega\, t}\ dt \\ &= \int_{0}^{\infty} e^{-at}(1)\ e^{-j\,\omega\, t}\ dt \end{aligned}$$

Fig. 5.11 Example P.5.7



$$
\begin{aligned}
&= \int_0^\infty e^{-(a+j\omega)t}\, dt \\
&= \{\text{integration by change of variable}\} \\
&= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_0^\infty \\
&= -\frac{1}{a+j\omega}(0 - \cancel{e^0}^{1}) \\
\therefore \\
X(\omega) &= \frac{1}{a+j\omega}, \quad \in \mathbb{C}
\end{aligned}
$$

Modulus $|X(\omega)|$ and argument $\theta(\omega) = \arg X(\omega)$ plots of complex function $X(\omega)$ (i.e. Bode plot) are derived as,

(a) $|X(\omega)|$: real and imaginary parts of an inverse complex function may be derived as

$$
|X(\omega)| = \left|\frac{1}{a+j\omega}\ \ \frac{a-j\omega}{a-j\omega}\right| = \left|\frac{a-j\omega}{a^2+\omega^2}\right| = \left|\frac{a}{a^2+\omega^2} - j\ \frac{\omega}{a^2+\omega^2}\right|
$$

$$
\{\text{note negative } \Re(X(\omega)) > 0 \text{ and } \Im(X(\omega)) < 0 \text{ parts}\}
$$

$$
= \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{\cancel{a^2+\omega^2}}{(a^2+\omega^2)^{\cancel{2}}}} = \frac{1}{\sqrt{a^2+\omega^2)}}
$$

or, alternatively

$$
|X(\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{1}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}
$$

It is sufficient to evaluate this even function in only a few important points as

$$
\omega = 0 \ \ \therefore \ \ |X(\omega)| = \frac{1}{a}
$$

$$
\omega = \pm a \ \ \therefore \ \ |X(\omega)| = \frac{1}{a\sqrt{2}}
$$

$$
\lim_{\omega \to \pm\infty} |X(\omega)| = 0
$$

then, its graph is sketched in Fig. 5.12 (top).

(b) $\theta(\omega) = \arg X(\omega)$: as $\Re(X(\omega)) > 0$ and $\Im(X(\omega)) < 0$ the argument $\theta(\omega)$ is in the fourth quadrant, thus negative.

$$
\theta(\omega) \stackrel{\text{def}}{=} \arctan \frac{\Im(X(\omega))}{\Re(X(\omega))} = \arctan \frac{-\dfrac{\omega}{\cancel{\sqrt{a^2+\omega^2}}}}{\dfrac{a}{\cancel{\sqrt{a^2+\omega^2}}}} = \operatorname{atan2}(-\omega, a) = -\arctan \frac{\omega}{a}
$$

Fig. 5.12 Example P.5.7

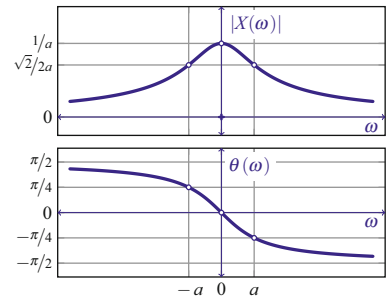
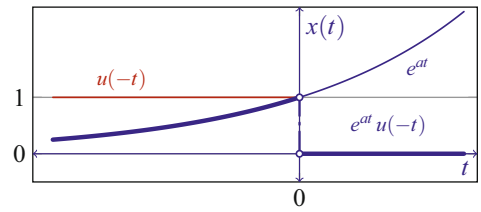


Fig. 5.13 Example P.5.8



It is sufficient to evaluate this odd function in only a few important points as

$$\omega = 0 \;\; \therefore \;\; \theta(0) = 0$$

$$\omega = +a \;\; \therefore \;\; \theta(a) = -\arctan(1) = -\frac{\pi}{4}$$

$$\omega = -a \;\; \therefore \;\; \theta(-a) = -\arctan(-1) = \frac{\pi}{4}$$

$$\lim_{\omega \to \infty} \theta(\omega) = -\frac{\pi}{2}$$

$$\lim_{\omega \to -\infty} \theta(\omega) = \frac{\pi}{2}$$

then, its graph is sketched in Fig. 5.12 (bottom).

5.8. Sketch of exponential $x(t) = e^{at}\, u(-t)$ functions product is in Fig. 5.13. By definition, Fourier integral of $x(t), a > 0$, is

$$X(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\, t}\, dt = \int_{-\infty}^{\infty} e^{at} u(-t)\, e^{-j\,\omega\, t}\, dt = \int_{-\infty}^{0} e^{at}\, e^{-j\,\omega\, t}\, dt = \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt$$

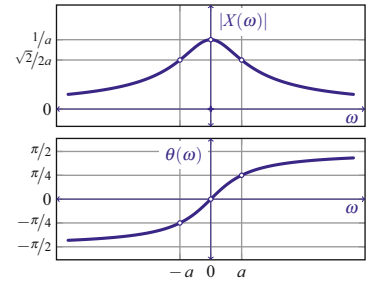
$$= \{\text{integration by change of variable}\}$$

$$= \frac{1}{a - j\omega}\, e^{(a-j\omega)t}\Big|_{-\infty}^{0} = \frac{1}{a - j\omega}(\cancel{e^{0}}^{1} - 0)$$

$\therefore$

$$X(\omega) = \frac{1}{a - j\omega}, \quad \in \mathbb{C}$$

Fig. 5.14 Example P.5.8



Modulus $|X(\omega)|$ and argument $\theta(\omega) = \arg X(\omega)$ plots of complex function $X(\omega)$ (i.e. Bode plot) are derived as,

(a) $|X(\omega)|$: real and imaginary parts of an inverse complex function may be derived as

$$|X(\omega)| = \left| \frac{1}{a - j\omega} \quad \frac{a + j\omega}{a + j\omega} \right| = \left| \frac{a + j\omega}{a^2 + \omega^2} \right| = \left| \frac{a}{a^2 + \omega^2} + j \frac{\omega}{a^2 + \omega^2} \right|$$

$$\{\text{note negative } \Re(X(\omega)) > 0 \text{ and } \Im(X(\omega)) > 0 \text{ parts}\}$$

$$= \sqrt{\frac{a^2}{(a^2 + \omega^2)^2} + \frac{\omega^2}{(a^2 + \omega^2)^2}} = \sqrt{\frac{\cancel{a^2 + \omega^2}}{(a^2 + \omega^2)\cancel{^2}}} = \frac{1}{\sqrt{a^2 + \omega^2)}}$$

or, alternatively

$$|X(\omega)| = \left| \frac{1}{a - j\omega} \right| = \frac{1}{|a - j\omega|} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

Module of a complex and its conjugate number are same, compare $|X(\omega)|$ in Fig. 5.12 (top) and Fig. 5.14 (top).

(b) $\theta(\omega) = \arg X(\omega)$: as $\Re(X(\omega)) > 0$ and $\Im(X(\omega)) > 0$ the argument $\theta(\omega)$ is in the first quadrant, thus positive.

$$\theta(\omega) \stackrel{\text{def}}{=} \arctan \frac{\Im(X(\omega))}{\Re(X(\omega))} = \arctan \frac{\omega}{\frac{\cancel{a^2 + \omega^2}}{a}} = \text{atan2}\,(\omega, a) = \arctan \frac{\omega}{a}$$

It is sufficient to evaluate this odd function in only a few important points as

$$\omega = 0 \quad \therefore \quad \theta(0) = 0$$

$$\omega = +a \quad \therefore \quad \theta(a) = \arctan(1) = \frac{\pi}{4}$$

$$\omega = -a \quad \therefore \quad \theta(-a) = \arctan(-1) = -\frac{\pi}{4}$$

$$\lim_{\omega \to \infty} \theta(\omega) = \frac{\pi}{2}$$

$$\lim_{\omega\to-\infty}\theta(\omega)=-\frac{\pi}{2}$$

then, its graph is sketched in Fig. 5.14 (bottom).

5.9. Given function $x(t)=e^{-a|t|}$ may be written as

$$x(t)=e^{-a|t|}=\begin{cases}e^{-at}, & t>0\\ 1, & t=0\\ e^{at}, & t<0\end{cases}$$

as illustrated in Fig. 5.15 (left). Fourier transformation is derived by definition as

$$\begin{aligned}X(\omega)&\overset{\text{def}}{=}\int_{-\infty}^{\infty}x(t)\,e^{-j\,\omega\,t}\,dt=\int_{-\infty}^{\infty}e^{-a|t|}\,e^{-j\,\omega\,t}\,dt=\int_{-\infty}^{0}e^{at}\,e^{-j\,\omega\,t}\,dt+\int_{0}^{\infty}e^{-at}\,e^{-j\,\omega\,t}\,dt\\&=\{\text{see P.5.8 and P.5.7}\}\\&=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=\frac{\cancel{a+j\omega}+\cancel{a-j\omega}}{a^2+\omega^2}\end{aligned}$$

$$\therefore$$

$$X(\omega)=\frac{2a}{a^2+\omega^2}\quad\therefore\quad X(\omega)\in\mathbb{R}\quad\Rightarrow\begin{cases}|X(\omega)| &= X(\omega)\\ \theta(\omega) &= 0^\circ\end{cases}$$

Being a real function $X(\omega)$ its argument $\theta(\omega)=\arg X(\omega)=0^\circ$, i.e. it is a constant. It is sufficient to evaluate $X(\omega)$ function in only a few important points as

$$\omega=0\quad\therefore\quad|X(\omega)|=\frac{2}{a}$$

$$\omega=\pm a\quad\therefore\quad|X(\omega)|=\frac{1}{a}$$

$$\lim_{\omega\to\pm\infty}|X(\omega)|=0$$

then, its graph is sketched in Fig. 5.15 (right).

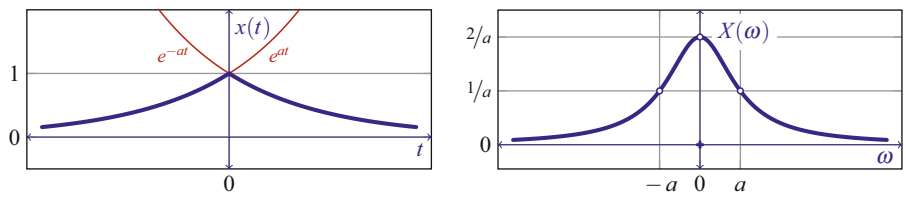


Fig. 5.15 Example P.5.9

5.10. Fourier transformation of $\Pi(t)$ function $x(t) = a\,\Pi(t/\tau)$ where $t \in (-\tau/2, \tau/2)$ is

$$
\begin{aligned}
X(\omega) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt = \int_{-\tau/2}^{\tau/2} a\, e^{-j\omega t}\, dt \\
&= \{\text{integration by change of variable}\} \\
&= -\frac{a}{j\omega}\, e^{-j\omega t}\,\Big|_{-\tau/2}^{\tau/2} = -\frac{a}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) \\
&= \frac{a}{j\omega}\,\frac{2}{2}\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right) = \frac{2a}{\omega}\,\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \\
&= \frac{2a}{\omega}\,\frac{\tau/2}{\tau/2}\,\sin\left(\frac{\omega\tau}{2}\right) = 2a\,\frac{\tau}{2}\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \\
&= a\,\tau\,\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$

In conclusion, $\Pi(t)$ and $\operatorname{sinc}(\omega)$ are dual functions in respect to Fourier transformation, Fig. 5.16.

5.11. Fourier transform of the triangle function $x(t) = \Lambda(t/2\tau)$ where $t \in (-\tau, \tau)$ may be performed by exploring derivative properties of Fourier transform of function derivatives (Fig. 5.17).

Reminder: Given $f(t)$ and its Fourier transformation $F(\omega)$, derivative property of FT is

$$
\begin{aligned}
\mathcal{F}\left(f'(x)\right) &= j\omega\, F(\omega) \\
\mathcal{F}\left(f''(x)\right) &= (j\omega)^2\, F(\omega) = -\omega^2\, F(\omega) \\
\mathcal{F}\left(f^{(3)}(x)\right) &= (j\omega)^3\, F(\omega) = -j\omega^3\, F(\omega) \\
&\cdots
\end{aligned}
$$

or, in general

$$
f^{(n)}(t) \stackrel{\text{FT}}{\rightarrow} (-j\omega)^n F(\omega)
$$

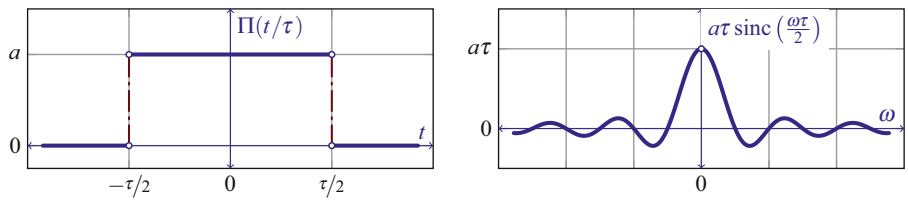
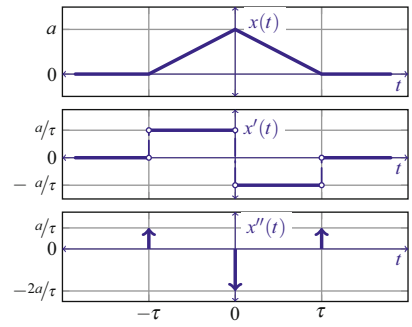


Fig. 5.16 Example P.5.10

Fig. 5.17 Example P.5.11



Derivatives of a triangle function may be deduced by graph inspection method.

(a) First derivative $x'(t)$ of a piecewise linear triangle function $x(t)$ may be deduced for each interval separately, Fig. 5.17 (top) and (middle). Horizontal intervals of $x(t)$ are constant, thus their derivatives equal zero. Two slope intervals are constant, thus by definition of tangent function in right triangle, as $a/(\pm\tau)$, i.e.

$$\frac{dx}{dt} = \frac{a}{\tau}; \quad x \in (-\tau, 0)$$
$$\frac{dx}{dt} = -\frac{a}{\tau}; \quad x \in (0, \tau)$$

(b) Second derivatives $x''(t)$ indicate points where $x'(t)$ change. Evidently, see Fig. 5.17 (middle), there are instantaneous changes of $x'(t)$ at $t = (-\tau, 0, \tau)$. An instantaneous change is modelled by Dirac function. Sharp step from zero to a/τ at $(t = -\tau)$ generates positive Delta function, sharp step from zero to $-a/\tau$ at $t = \tau$ generates negative Delta function, and sharp step from a/τ to $-a/\tau$ at $t = 0$ generates negative $-2a/\tau$ Delta function.

By inspection of Fig. 5.17 (bottom) it follows that

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{a}{\tau}\,\delta(t+\tau) - \frac{2a}{\tau}\,\delta(t) + \frac{a}{\tau}\,\delta(t-\tau) \\ &= \frac{a}{\tau}\,(\delta(t+\tau) - 2\,\delta(t) + \delta(t-\tau)) \end{aligned} \tag{5.9}$$

Fourier transformations of Dirac delta functions are already derived in A.5.1 and A.5.2. Then, by knowing FT of each Dirac term in (5.9) and by applying derivative property of FT it follows that

$$(j\omega)^2\, X(\omega) = \mathscr{F}\left(\frac{d^2x}{dt^2}\right) = \frac{a}{\tau}\left(e^{j\omega\tau} - 2 + e^{-j\omega\tau}\right)$$

$$\therefore$$

$$X(\omega) = \frac{a}{-\omega^2\tau}\Big(\underbrace{e^{j\omega\tau} + e^{-j\omega\tau}}_{2\cos(\omega\tau)} - 2\Big) = \frac{2a}{\omega^2\tau}\Big(\underbrace{1 - \cos\omega\tau}_{2\sin^2(\omega\tau/2)}\Big)$$

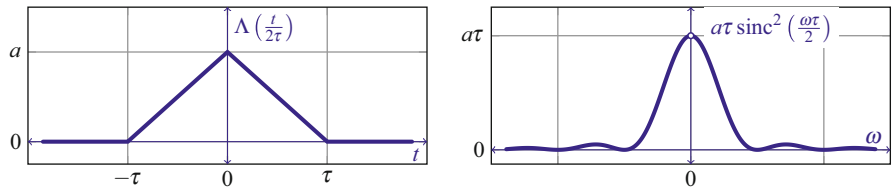
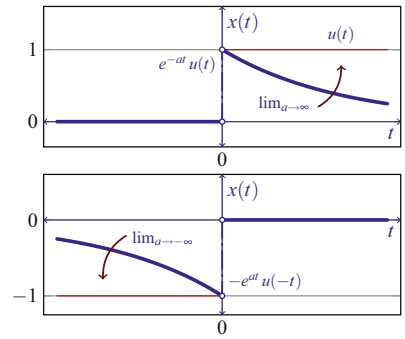


Fig. 5.18 Example P.5.11

Fig. 5.19 Example P.5.12



$$= \frac{\tau}{\tau}\,\frac{2a}{\omega\omega\tau}\;2\sin^2\left(\frac{\omega\tau}{2}\right) = a\tau\;\;\frac{\sin(\omega\tau/2)}{\omega\tau/2}\;\;\frac{\sin(\omega\tau/2)}{\omega\tau/2}$$

$$= a\tau\;\;\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)\;\;\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$= a\tau\;\;\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$

In conclusion, $\Lambda(t)$ and $\mathrm{sinc}^2[\omega]$ are dual functions in respect to Fourier transformation, Fig. 5.18.

5.12. Sign function $\mathrm{sign}(t)$ is non–convergent for $t \to \infty$, and by consequence direct application of Fourier integral also produces non–converging result. Some of the possible techniques that may be used are as follows (Fig. 5.19).

Method 1: sign function may be expressed as

$$\mathrm{sign}(t) = u(t) - u(-t)$$

In addition, $e^0 = 1$, which can be also written as

$$\lim_{a\to 0} e^{\pm at} = 1$$

and therefore it is also valid to write

$$u(t) = (1)\,u(t) = \lim_{a=0} e^{-at}\,u(t)$$

$$-u(-t) = (-1)\,u(-t) = -\lim_{a=0} e^{at}\,u(-t)$$

By doing so, see Fig. 5.19, sign (t) is replaced with the equivalent $x(t)$ that is convergent, as

$$\begin{aligned} x(t) = \text{sign}\,(t) = u(t) - u(-t) &= \lim_{a\to 0}\left(e^{-at}\,u(t) - e^{at}\,u(-t)\right) \\ &= \{\text{see A.5.7 and A.5.8}\} \\ \therefore \\ X(\omega) = \mathscr{F}\big(x(t)\big) &= \lim_{a\to 0}\left(\frac{1}{\cancel{a}^{0} + j\omega} - \frac{1}{\cancel{a}^{0} - j\omega}\right) = \frac{1}{j\omega} + \frac{1}{j\omega} \\ &= \frac{2}{j\omega} \end{aligned}$$

Method 2: exploit the derivative property of Fourier transform as

$$\begin{aligned} x(t) = \text{sign}\,(t) = 2u(t) - 1 \quad &\{\text{see A.2.11 and A.5.1}\} \\ \therefore \\ \frac{dx(t)}{dt} = 2\,\delta(t) \quad \overset{FT}{\longrightarrow} \quad j\omega\; X(\omega) = 2 \quad &\therefore \quad X(\omega) = \frac{2}{j\omega} \end{aligned}$$

5.13. Step function $u(t)$ is nom–convergent for $t \to \infty$, and by consequence direct application of Fourier integral also produces non–converging result. However, see A.2.30 and A.5.12,

$$\begin{aligned} x(t) = u(t) &= \frac{1 + \text{sign}\,(t)}{2} \\ \therefore \\ X(\omega) = \mathscr{F}\left[\frac{1}{2} + \frac{\text{sign}\,(t)}{2}\right] &= \{\mathscr{F}(\text{ constant }) + \mathscr{F}(\text{sign}\,(t)\,\}) \\ &= \mathscr{F}\left(\frac{1}{2}\right) + \frac{1}{2}\mathscr{F}\big(\text{sign}\,(t)\big) = \cancel{2}\pi\;\frac{1}{\cancel{2}}\,\delta(\omega) + \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{j\omega} \\ &= \pi\,\delta(t) + \frac{1}{j\omega} \end{aligned}$$

5.14. Derivative of Gaussian function is

$$x(t) = e^{-t^2} \quad \therefore \quad x'(t) = -2t\,e^{-t^2} = -2t\,x(t) \tag{5.10}$$

therefore, FT of left and right sides in (5.10) is

$$\mathscr{F}\left(x'(t)\right) = -2\mathscr{F}\big(tx(t)\big) \tag{5.11}$$

after substituting FT derivative property $\mathscr{F}\left(x'(t)\right) = j\omega\; X(\omega)$ in (5.11) it follows that

$$j\omega\; X(\omega) = -2\mathscr{F}\big(tx(t)\big) \tag{5.12}$$

thus it is necessary to derive $\mathscr{F}\big(t\,x(t)\big)$. By definition,

$$X(\omega) = \mathscr{F}\big(x(t)\big) = \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\,t}\, dt$$

$$\therefore \quad \{\text{left and right side derivative, note that } x(t) \neq f(\omega)\}$$

$$\frac{d}{d\omega} X(\omega) = \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\,t}\, dt = \int_{-\infty}^{\infty} x(t)\, \frac{d}{d\omega} e^{-j\,\omega\,t}\, dt$$

$$= -j\omega \int_{-\infty}^{\infty} t\, x(t)\, e^{-j\,\omega\,t}\, dt \stackrel{\text{def}}{=} -j\omega\, \mathscr{F}\big(t\, x(t)\big)$$

$$\therefore$$

$$\mathscr{F}\big(t\, x(t)\big) = \frac{1}{-j\omega} \frac{j}{j} X'(\omega) = \frac{j}{\omega} X'(\omega) \tag{5.13}$$

Substituting (5.13) into (5.12) results in a first order differential equation with constant coefficients relative to $X(\omega)$ as

$$j\omega\, X(\omega) = -2\, \frac{j}{\omega} X'(\omega)$$

$$X'(\omega) = -\frac{\omega^2}{2} X(\omega) \quad \{X(\omega) = y\}$$

This differential equation is solved by the separation of variables method as

$$\frac{dy}{d\omega} = -\frac{\omega}{2}\, y \quad \Rightarrow \quad \int \frac{dy}{y} = -\frac{1}{2} \int \omega\, d\omega \quad \Rightarrow \quad \ln y - \ln C = -\frac{1}{4}\omega^2 \quad \therefore \quad \underline{y = C\, e^{-\omega^2/4}}$$

Return to $X(\omega)$,

$$X(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{-x^2}\, e^{-j\,\omega\,t}\, dt = C\, e^{-\frac{\omega^2}{4}}$$

Simplest initial condition is $X(0) = C$, so that $e^0 = 1$ and

$$C = \int_{-\infty}^{\infty} e^{-x^2}\, dt = \{\text{see Vol. II}\} = \sqrt{\pi}$$

In conclusion,

$$X(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$$

5.15. Fourier transform of sinc (t) function may be derived as as follows

$$X(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\,t}\, dt = \int_{-\infty}^{\infty} \frac{\sin(\omega_0\, t)}{\omega_0\, t}\, e^{-j\,\omega\,t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\omega_0\, t}\, \sin(\omega_0\, t) \Big[\cos(\omega\, t) - j\, \sin(\omega\, t)\Big]\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\omega_0\, t}\ \sin(\omega_0\, t)\ \cos(\omega\, t)\ dt - j \int_{-\infty}^{\infty} \frac{1}{\omega_0\, t}\ \sin(\omega_0\, t)\ \sin(\omega\, t)\ dt$$
$$= I_1 - j\, I_2$$

See Vol. II where it is shown that

$$\int_0^{\infty} e^{-\omega_0\, t\, y}\, dy = \frac{1}{\omega_0\, t} \tag{5.14}$$

That is to say, the two integrals I_1 and I_2 are solved separately as follows. First, solution to I_1

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{\omega_0\, t}\ \sin(\omega_0\, t)\ \cos(\omega\, t)\ dt$$
$$= \{\text{substitute (5.14) and use trig identity for product, } as\ \}$$
$$= \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\omega_0\, t\, y}\, dy\ \frac{1}{2}\left(\sin\big((\omega_0 + \omega)\, t\big] + \sin[(\omega_0 - \omega)\, t\big)\right) dt$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\omega_0\, t\, y}\, dy \left(\sin\big((\omega_0 + \omega)\, t\big] + \sin[(\omega_0 - \omega)\, t\big)\right) dt$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} dy \int_0^{\infty} e^{-\omega_0\, y\, t} \sin\big((\omega + \omega_0)\, t\big)\, dt - \frac{1}{2} \int_{-\infty}^{\infty} dy \int_0^{\infty} e^{-\omega_0\, y\, t} \sin\big((\omega - \omega_0)\, t\big)\, dt$$
$$= \{\text{see: Vol. II }\} =$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\omega + \omega_0}{(\omega + \omega_0)^2 + (\omega_0\, y)^2}\right] dy - \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\omega - \omega_0}{(\omega - \omega_0)^2 + (\omega_0\, y)^2}\right] dy$$
$$\{\text{see: Vol. II, as well as } \omega,\ \omega_0 \neq f(y) \text{ i.e. they are constants relative to } y\ \}$$
$$= \frac{\pi}{2\omega_0} \Big[\text{sign}\,(\omega + \omega_0) - \text{sign}\,(\omega - \omega_0)\Big]$$

The last result defines $\Pi_{(\omega/2\omega_0)}$ as illustrated in Fig. 5.20, which is expected as dual function of sinc (t) function.

Integral I_2 is solved by following the same idea as for I_1,

$$I_2 = \int_{-\infty}^{\infty} \frac{1}{\omega_0\, t}\ \sin(\omega_0\, t)\ \sin(\omega\, t)\ dt$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\omega_0\, t\, y}\, dy \left(\cos\big((\omega - \omega_0)\, t\big) - \cos\big((\omega + \omega_0)\, t\big)\right) dt$$
$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} dy \int_0^{\infty} e^{-\omega_0\, y\, t} \cos\big((\omega - \omega_0)\, t\big)\, dt - \int_{-\infty}^{\infty} dy \int_0^{\infty} e^{-\omega_0\, y\, t} \cos\big((\omega + \omega_0)\, t\big)\, dt\right]$$
$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} \frac{\omega_0\, y}{(\omega - \omega_0)^2 + (\omega_0\, y)^2}\, dy - \int_{-\infty}^{\infty} \frac{\omega_0\, y}{(\omega + \omega_0)^2 + (\omega_0\, y)^2}\, dy\right]$$

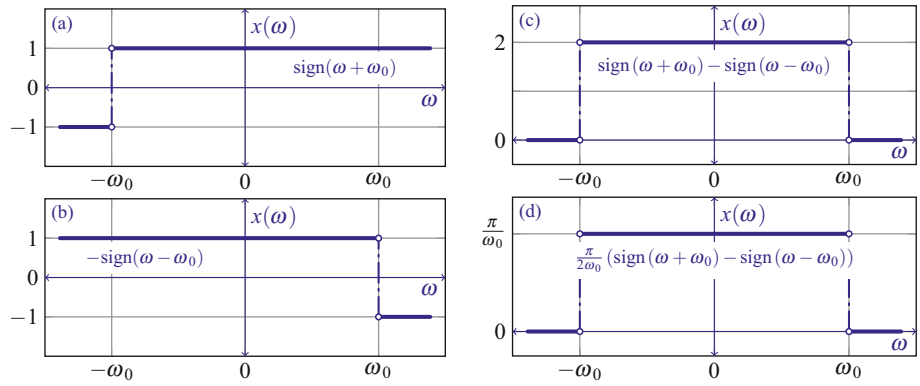


Fig. 5.20 Example P.5.15

$$\left\{\begin{array}{l} x \;\; = (\omega \pm \omega_0)^2 + (\omega_0\, y)^2 \;\;\therefore\;\; dx = 2\omega_0^2\, y\, dy \;\;\therefore\;\; \omega_0\, y\, dy = \dfrac{1}{2\omega_0}\, dx \\ \\ \quad \therefore\;\; \dfrac{1}{2\omega_0}\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x} = \frac{1}{2\omega_0}\, \ln x \,\Big|_{-\infty}^{\infty} \end{array}\right\}$$

$$= \frac{1}{4\omega_0}\left[\, \ln\left[(\omega-\omega_0)^2 + (\omega_0\, y)^2\right]\Big|_{-\infty}^{\infty} - \ln\left[(\omega+\omega_0)^2 + (\omega_0\, y)^2\right]\Big|_{-\infty}^{\infty}\right]$$

$$= \frac{1}{4\omega_0}\left[\, \ln\left[(\omega-\omega_0)^2 + (\omega_0\, y)^2\right]\Big|^{\infty} - \ln\left[(\omega-\omega_0)^2 + (\omega_0\, y)^2\right]\Big|_{-\infty}\right.$$

$$\left. - \ln\left[(\omega+\omega_0)^2 + (\omega_0\, y)^2\right]\Big|^{\infty} + \ln\left[(\omega+\omega_0)^2 + (\omega_0\, y)^2\right]\Big|_{-\infty}\right]$$

$$\left\{\ln a + \ln b = \ln(ab);\;\; \ln a - \ln b = \ln\frac{a}{b};\;\; \ln a^b = b\ln a\right\}$$

$$= \frac{1}{4\omega_0}\left[\, \ln\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\Big|^{\infty} - \ln\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\Big|_{-\infty}\right]$$

$$= \frac{1}{4\omega_0}\left\{\lim_{y\to\infty}\left[\ln\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\right] - \lim_{y\to-\infty}\left[\ln\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\right]\right\}$$

$$= \frac{1}{4\omega_0}\left\{\ln\left[\lim_{y\to\infty}\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\right] - \ln\left[\lim_{y\to-\infty}\frac{(\omega-\omega_0)^2 + (\omega_0\, y)^2}{(\omega+\omega_0)^2 + (\omega_0\, y)^2}\right]\right\} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \frac{1}{4\omega_0}\left\{\ln\left[\lim_{y\to\infty}\frac{2\omega_0^2\, y}{2\omega_0^2\, y}\right] - \ln\left[\lim_{y\to-\infty}\frac{2\omega_0^2\, y}{2\omega_0^2\, y}\right]\right\} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \frac{1}{4\omega_0}\left\{\ln\left[\lim_{y\to\infty}\frac{2\omega_0^2}{2\omega_0^2}\right] - \ln\left[\lim_{y\to-\infty}\frac{2\omega_0^2}{2\omega_0^2}\right]\right\}$$

$$\overset{\text{l'H}}{=} \frac{1}{4\omega_0}\left[\ln(1) - \ln(1)\right] = 0$$

In conclusion,

$$X(\omega) = \int_{-\infty}^{\infty} \frac{\sin(\omega_0\, t)}{\omega_0\, t}\, e^{-j\,\omega\, t}\, dt = I_1 + j\,\cancelto{0}{I_2} = I_1 = \frac{\pi}{2\omega_0}\Big[\text{sign}\,(\omega+\omega_0) - \text{sign}\,(\omega-\omega_0)\Big]$$

5.2 Composite Functions

5.16. Given $x(t)$ may be decomposed into the sum of two $\Pi\,(t)$ functions, $x_1(t)$ and $x_2(t)$ as illustrated in Fig. 5.21 (left), where

$$x_1(t) = \Pi\left(\frac{t}{4}\right) \quad \text{and} \quad x_2(t) = \Pi\left(\frac{t}{2}\right)$$

$$\therefore$$

$$x(t) = \Pi\left(\frac{t}{4}\right) + \Pi\left(\frac{t}{2}\right)$$

the total Fourier transformation is therefore the sum of two FT, see P.5.10, as

$$X(\omega) = \mathscr{F}\left(\Pi\left(\frac{t}{4}\right)\right) + \mathscr{F}\left(\Pi\left(\frac{t}{2}\right)\right) = 4\,\text{sinc}\,(2\omega) + 2\,\text{sinc}\,(\omega)$$

5.17. Using graphical derivation method and derivative properties of FT it follows that,

$$x(t) = -2\,\text{sign}\,(t-2) \;\;\therefore\;\; x'(t) = -4\delta(t-2) \;\;\therefore\;\; j\omega\, X(\omega) = -4e^{-2j\omega}$$

$$\therefore$$

$$X(\omega) = -\frac{4}{j\omega}\, e^{-2j\omega}$$

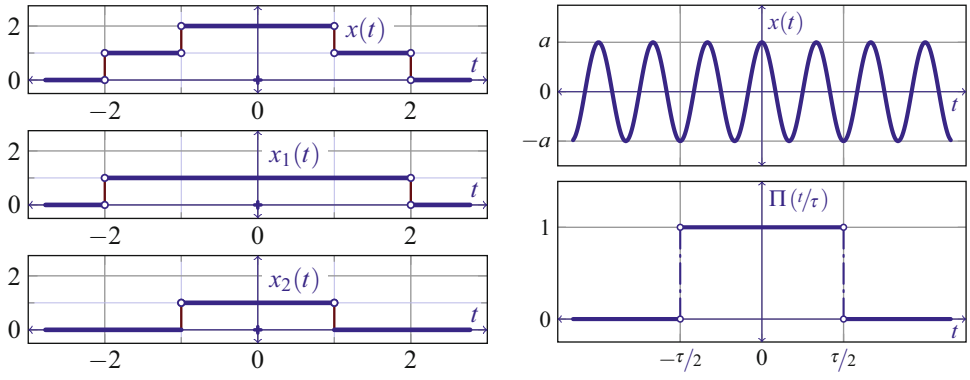


Fig. 5.21 Examples P.5.16, P.5.18

5.18. Given RF pulse $h(t)$ may be decomposed into the product of $x(t) = a\cos(j\,\omega_0\,t)$ and $\Pi(t/\tau)$ functions, Fig. 5.21 (right), i.e.

$$h(t) = a\cos(j\,\omega_0\,t)\;\Pi\,(t/\tau)$$

where, $\omega_0 = 2\pi\;f_0$ is the RF sine frequency, and τ is duration of RF pulse.

By using Euler's formula it follows that,

$$x(t) = a\cos(\omega_0\,t) = \frac{a}{2}\big(\exp(j\,\omega_0\,t) + \exp(-j\,\omega_0\,t)\big)$$

then, Fourier transformation is derived by applying the frequency shifting and time scaling properties,

Reminder:

Frequency shifting property: $e^{-j\,\omega_0\,t}\,x(t) \overset{\text{FT}}{\rightarrow} X(\omega - \omega_0)$

Time scaling property: $x(at) \overset{\text{FT}}{\rightarrow} \dfrac{1}{|a|}X\left(\dfrac{\omega}{a}\right)$

that is to say

$$\Pi\,(t) \overset{\text{FT}}{\rightarrow} \text{sinc}\,(t) \;\;\therefore\;\; \Pi\,(t)\exp(j\,\omega_0\,t) \overset{\text{FT}}{\rightarrow} \text{sinc}\,(\omega - \omega_0)$$

$$\therefore$$

$$h(t) = \frac{a}{2}\Pi\,(t/\tau)\exp(j\,\omega_0\,t) + \frac{a}{2}\Pi\,(t/\tau)\exp(-j\,\omega_0\,t)$$

$$\therefore$$

$$H(\omega) = \mathscr{F}\big(h(t)\big) = \frac{a\tau}{2}\left(\text{sinc}\,(\omega)\,\big(\tau(\omega - \omega_0)\big) + \text{sinc}\,(\omega)\,\big(\tau(\omega + \omega_0)\big)\right)$$

That is to say, if $\omega_0\,\tau \gg 1$ then, see Fig. 5.22

$$H(\omega) = \begin{cases} \dfrac{a\tau}{2}\text{sinc}\,(\omega)\,\big(\tau(\omega - \omega_0)\big) & ;\;\; \omega > 0 \\ 0 & ;\;\; \omega = 0 \\ \dfrac{a\tau}{2}\text{sinc}\,(\omega)\,\big(\tau(\omega + \omega_0)\big) & ;\;\; \omega < 0 \end{cases}$$

Fig. 5.22 Example P.5.18

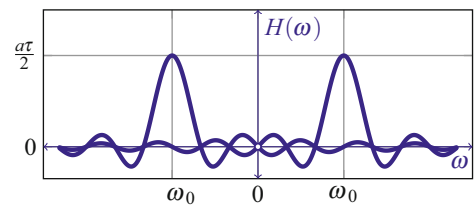
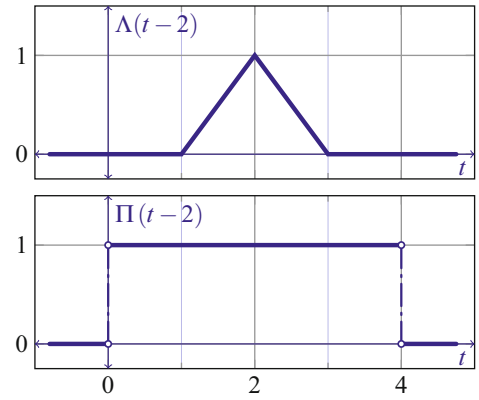


Fig. 5.23 Example P.5.19



5.19. Given function $x(t)$ may be decomposed into the sum of port and triangle funcitons functions, Fig. 5.23, i.e.

$$x(t) = \Pi\left(t/4 - 2\right) + \Lambda\left(t/2 - 2\right)$$

then, Fourier transformation is derived by applying the time shifting and scaling properties as

$$\begin{aligned}\mathscr{F}\big(x(t)\big) &= \mathscr{F}\left(\Pi\left(t/4 - 2\right)\right) + \mathscr{F}\left(\Lambda\left(t/2 - 2\right)\right)\\ &= 4e^{-j\,2\omega}\,\mathrm{sinc}\left(4\omega\right) + 2e^{-j\,2\omega}\,\mathrm{sinc}^2\left(2\omega\right)\end{aligned}$$

5.20. Given function $x(t)$ in Fig. 5.6 may be decomposed into the sum of two port functions, as

$$x(t) = \Pi\left(t/2 - 1\right) + \Pi\left(t/2 + 3\right)$$

that is to say

$$\begin{aligned}\mathscr{F}\big(x(t)\big) &= \mathscr{F}\left(\Pi\left(t/2 - 1\right)\right) + \mathscr{F}\left(\Pi\left(t/2 + 3\right)\right)\\ &= 2e^{-j\omega}\,\mathrm{sinc}\left(2\omega\right) + 2e^{j\,3\omega}\,\mathrm{sinc}\left(3\omega\right)\end{aligned}$$

5.21. Given $x(t)$ in Fig. 5.7, it may be decomposed into the sum of two triangle functions, as

$$x(t) = \Lambda\big(t/2 - 1\big) + \Lambda\big(t/2 + 3\big)$$

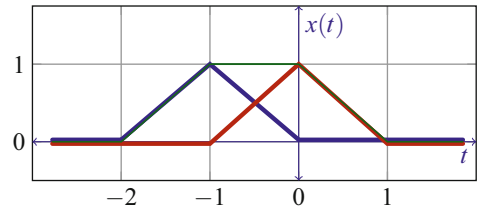
that is to say, after applying the time scaling and delay properties of FT,

$$\begin{aligned}\mathscr{F}\big(x(t)\big) &= \mathscr{F}\left(\Lambda\big(t/2 - 1\big)\right) + \mathscr{F}\left(\Lambda\big(t/2 + 3\big)\right)\\ &= 2e^{-j\omega}\,\mathrm{sinc}^2\left(2\omega\right) + 2e^{j\,3\omega}\,\mathrm{sinc}^2\left(3\omega\right)\end{aligned}$$

5.22. Given function $x(t)$ in Fig. 5.8 may be decomposed into the sum of two triangle functions, see Fig. 5.24, as

$$x(t) = \Lambda\big(t/2 - 1\big) + \Lambda\big(t/2\big)$$

Fig. 5.24 Example P.5.22



that is to say, after applying the time scaling and delay properties of FT,

$$\begin{aligned}\mathscr{F}\left(x(t)\right) &= \mathscr{F}\left(\Lambda\left(t/2-1\right)\right)+\mathscr{F}\left(\Lambda\left(t/2\right)\right)\\ &= 2e^{-j\omega}\,\mathrm{sinc}^2\left(2\omega\right)+2\mathrm{sinc}^2\left(2\omega\right)\\ &= 2\mathrm{sinc}^2\left(2\omega\right)\left(e^{-j\omega}+1\right)\end{aligned}$$

5.3 Fourier Series

5.23. Given $f(t)=a$, in $(-\pi,\pi)$ interval, coefficients of Fourier series are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}a\,dt=\frac{a}{\pi}\,t\,\Big|_{-\pi}^{\pi}=\frac{a}{\pi}2\pi=2a$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}a\cos(nt)\,dt=\frac{a}{n\pi}\,\sin(nt)\,\Big|_{-\pi}^{\pi}$$

$$=\frac{a}{n\pi}\left(\sin(n\pi)^{0}-\sin(-n\pi)^{0}\right)=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}a\sin(nt)\,dt=-\frac{a}{n\pi}\,\cos(nt)\,\Big|_{-\pi}^{\pi}$$

$$=-\frac{a}{n\pi}\left(\cos(n\pi)-\cos(n\pi)\right)=0$$

Therefore, Fourier series of $f(t)=a$ is

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nt)+\sum_{n=1}^{\infty}b_n\sin(nt)=\frac{2a}{2}+\sum_{n=1}^{\infty}0^{0}\cos(nt)+\sum_{n=1}^{\infty}0^{0}\sin(nt)$$

$$=a$$

That is to say, Fourier series of a constant is simply the constant itself.

5.24. Given sawtooth $f(t)=(a/\pi)\,t$, in $(-\pi,\pi)$ interval, coefficients of Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{a}{\pi}\,t\,dt = \frac{a}{2\pi^2}\,t^2\,\Big|_{-\pi}^{\pi} = \frac{a}{2\pi^2}\left(\pi^2 - (-\pi)^2\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{a}{\pi}\,t\cos(nt)\,dt = \{\text{see Vol. II}\}$$

$$= \frac{a}{\pi^2 n^2}\left(nt\,\sin(nt) + \cos(nt)\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{a}{\pi^2 n^2}\left(n\pi\,\cancelto{0}{\sin(n\pi)} + \cancel{\cos(n\pi)} - n(-\pi)\,\cancelto{0}{\sin(-n\pi)} - \cancel{\cos(-n\pi)}\right)$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{a}{\pi}\,t\sin(nt)\,dt = \{\text{see Vol. II}\}$$

$$= \frac{a}{n^2\pi^2}\left(-nt\cos(nt) + \sin(nt)\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{a}{n^2\pi^2}\left(-n\pi\cos(n\pi) + \cancelto{0}{\sin(n\pi)} + n(-\pi)\cos(-n\pi) - \cancelto{0}{\sin(-n\pi)}\right)$$

$$= -\frac{2\cancel{n}\cancel{\pi}a}{\cancel{n}^2\cancel{\pi}^{\cancel{2}}}\underbrace{\cos(n\pi)}_{(-1)^n} = -\frac{2a}{n\pi}(-1)^n$$

Therefore, Fourier series of sawtooth $f(t) = (a/\pi)\,t$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt) = \frac{0}{2} + \sum_{n=1}^{\infty} 0\cos(nt) + \sum_{n=1}^{\infty} -\frac{2a}{n\pi}(-1)^n\sin(nt)$$

$$= -\frac{2a}{n\pi}\sum_{n=1}^{\infty}(-1)^n\sin(nt)$$

$$= \frac{2a}{\pi}\sin(t) - \frac{2a}{2\pi}\sin(2t) + \frac{2a}{3\pi}\sin(3t) - \cdots$$

Note that sawtooth $f(t) = (a/\pi)\,t$function is *odd*, i.e. $f(x) = -f(-x)$, and therefore all $a_n = 0$. This is a general property of all odd functions. The sum of first five terms of the Fourier series $f(t)$ is already close approximation of ideal sawtooth function, see Fig. 5.25.

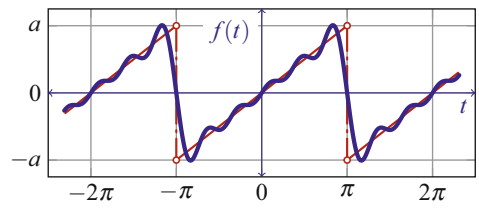


Fig. 5.25 Example P.5.24

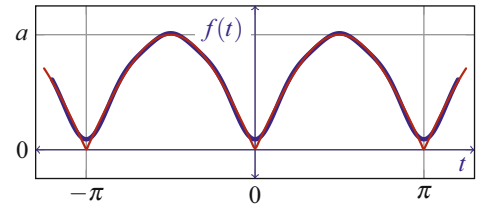
5.25. Given absolute sines $f(t) = |\sin t|$, in $(-\pi, \pi)$ interval, note that it is *an even function* and always positive as $|\sin t| \geq 0$ (a cosines is also even function). Thus, integrals may be calculated only in the positive $(0, \pi)$ interval then multiplied by two so that coefficients of Fourier series are

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin t|\,dt = \frac{2}{\pi}\int_0^{\pi} \sin t\,dt = \frac{2}{\pi}\,(-\cos t)\,\Big|_0^{\pi} \\
&= \frac{2}{\pi}(-\cos\pi + \cos 0) = \frac{4}{\pi} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin t|\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} \sin t\cos(nt)\,dt \\
&= \left\{\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)\right\} = \frac{1}{\pi}\int_0^{\pi} \sin(t+nt)\,dt + \frac{1}{\pi}\int_0^{\pi} \sin(t-nt)\,dt \\
&= \left\{t \pm nt = x \;\;\therefore\;\; (1\pm n)\,dt = dx \;\;\therefore\;\; dt = \frac{1}{1\pm n}\,dx\right\} \\
&= -\frac{1}{\pi}\frac{1}{1+n}\,\cos(t+nt)\,\Big|_0^{\pi} - \frac{1}{\pi}\frac{1}{1-n}\,\cos(t-nt)\,\Big|_0^{\pi} \\
&= -\frac{1}{\pi}\frac{1}{1+n}\Big(\underbrace{\cos\big((1+n)\pi\big)}_{(-1)^n} - \underbrace{\cos\big((1+n)\cdot 0\big)}_{=1}\Big) - \frac{1}{\pi}\frac{1}{1-n}\Big(\underbrace{\cos\big((1-n)\pi\big)}_{(-1)^n} - \underbrace{\cos\big((1-n)\cdot 0\big)}_{=1}\Big) \\
&= -\frac{1}{\pi}\frac{(-1)^n+1}{1+n} - \frac{1}{\pi}\frac{(-1)^n+1}{1-n} = \frac{2}{\pi}\frac{(-1)^n+1}{(1+n)(1-n)}, \quad (n \geq 2) \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin t|\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} \sin t\sin(nt)\,dt \\
&= \left\{\sin x\sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\right\} = \frac{1}{\pi}\int_0^{\pi} \cos(t-nt)\,dt - \frac{1}{\pi}\int_0^{\pi} \cos(t+nt)\,dt \\
&= \left\{t \pm nt = x \;\;\therefore\;\; (1\pm n)\,dt = dx \;\;\therefore\;\; dt = \frac{1}{1\pm n}\,dx\right\} \\
&= \frac{1}{\pi}\frac{1}{1-n}\,\sin(t-nt)\,\Big|_0^{\pi} - \frac{1}{\pi}\frac{1}{1+n}\,\sin(t+nt)\,\Big|_0^{\pi} \\
&= \frac{1}{\pi}\frac{1}{1-n}\Big(\underbrace{\sin\big((1-n)\pi\big)}_{=0} - \underbrace{\sin\big((1-n)\cdot 0\big)}_{=0}\Big) - \frac{1}{\pi}\frac{1}{1+n}\Big(\underbrace{\sin\big((1+n)\pi\big)}_{=0} - \underbrace{\sin\big((1+n)\cdot 0\big)}_{=0}\Big) \\
&= 0, \quad (n \geq 2)
\end{aligned}
$$

Therefore, Fourier series of absolute sines $f(t) = |\sin t|$ is

$$
\begin{aligned}
f(t) &= \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2}{\pi}\frac{(-1)^n+1}{(1+n)(1-n)}\cos(nt) + \sum_{n=2}^{\infty} 0\,\sin(nt) \\
&= \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2}{\pi}\frac{(-1)^n+1}{(1+n)(1-n)}\cos(nt)
\end{aligned}
$$

Fig. 5.26 Example P.5.25



$$= \frac{2}{\pi} - \frac{4}{3\pi}\cos(2t) + \frac{2}{\pi}\frac{\cancelto{0}{(-1)^3+1}}{(1+3)(1-3)}\cos(3t) - \frac{4}{15\pi}\cos(4t) + 0 - \frac{4}{35\pi}\cos(6t) + 0 - \cdots$$

Note that absolute sines $f(t) = |\sin t|$ function is *even*, i.e. $f(x) = f(-x)$, and therefore all $b_n = 0$. This is a general property of all even functions. The sum of first four terms of the Fourier series $f(t)$ is already close approximation of ideal absolute sines function, see Fig. 5.26.

5.26. Given sines cube $f(t) = \sin^3 t$, in $(-\pi, \pi)$ interval. An attempt to calculate coefficients of Fourier series by definition may turn out to be very long and tedious, as a_n and b_n integrals are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3 t\,dt = \{\text{see Vol. II}\} = \frac{1}{\pi}\left((-\cos t)\Big|_{-\pi}^{\pi} + \frac{1}{3}\cos^3 t\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\cancel{-\cos\pi} + \cancel{\cos(-\pi)} + \frac{1}{3}\left(\cancel{\cos^3\pi} - \cancel{\cos^3(-\pi)}\right)\right)$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3 t\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3 t\sin(nt)\,dt$$

which is doable by multiple methods but not always trivial to resolve.

Instead, by noting that $\sin^3 t$ is already trigonometric function, first it may be transformed by applying identities, as

$$\sin t\,\sin t = \frac{1}{2}\big(\cos(t-t) - \cos(t+t)\big) = \frac{1}{2}\big(1 - \cos(2t)\big)$$

$$\therefore$$

$$\sin^3 t = \sin^2 t\,\sin t = \frac{1}{2}\big(1-\cos(2t)\big)\sin t = \frac{1}{2}\sin t - \frac{1}{2}\cos(2t)\,\sin t$$

$$= \frac{1}{2}\sin t - \frac{1}{2}\left(\frac{1}{2}\big(\sin(2t+t) - \sin(2t-t)\big)\right)$$

$$= \frac{1}{2}\sin t - \frac{1}{4}\sin 3t + \frac{1}{4}\sin t$$

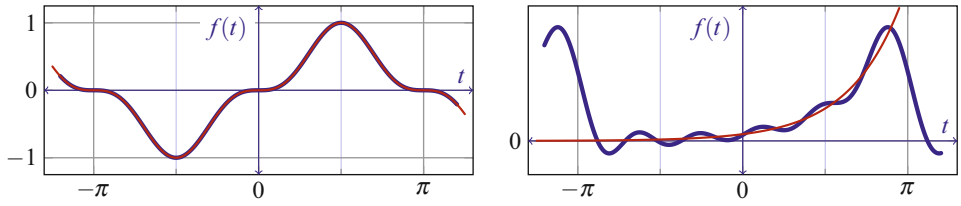


Fig. 5.27 Examples P.5.26, P.5.27

Therefore, Fourier series of sines cube $f(t) = \sin^3 t$ consists of only two non–zero terms , as

$$f(t) = \frac{3}{4}\,\sin t - \frac{1}{4}\,\sin 3t$$

and all the other terms are eliminated by zero coefficients. Which is to say that Fourier series does not have to be infinitely long. In addition, this finite series contains the only terms of sines that are needed to exactly create $\sin^3 t$ waveform, see Fig. 5.27 (left).

5.27. Given exponential function $f(t) = e^t$, in $(-\pi, \pi)$ interval, note that exponential function is neither odd nor even. Coefficients of Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t\,dt = \frac{1}{\pi}\,(e^t)\,\Big|_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t\cos(nt)\,dt = \{\text{see Vol. II}\}$$

$$= \frac{1}{\pi}\Big(\frac{n}{n^2+1}\,\underbrace{\sin(n\pi)}_{=0}\big(e^{\pi}+e^{-\pi}\big) + \frac{1}{n^2+1}\,\underbrace{\cos(n\pi)}_{(-1)^n}\big(e^{\pi}-e^{-\pi}\big)\Big)$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2+1}\big(e^{\pi}-e^{-\pi}\big)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t\sin(nt)\,dt = \{\text{see Vol. II}\}$$

$$= \frac{1}{\pi}\Big(-\frac{n}{n^2+1}\,\underbrace{\cos(n\pi)}_{(-1)^n}\big(e^{\pi}-e^{-\pi}\big) + \frac{1}{n^2+1}\,\underbrace{\sin(n\pi)}_{=0}\big(e^{\pi}+e^{-\pi}\big)\Big)$$

$$= -\frac{1}{\pi}\,\frac{n}{n^2+1}\,(-1)^n\,\big(e^{\pi}-e^{-\pi}\big)$$

Therefore, Fourier series of exponential function $f(t) = e^t$ is

$$f(t) = \frac{1}{2\pi}(e^{\pi}-e^{-\pi}) + \sum_{n=1}^{\infty}\frac{1}{\pi}\frac{(-1)^n}{n^2+1}\big(e^{\pi}-e^{-\pi}\big)\cos(nt) - \sum_{n=1}^{\infty}\frac{1}{\pi}\,\frac{n}{n^2+1}\,(-1)^n\,\big(e^{\pi}-e^{-\pi}\big)\,\sin(nt)$$

$$= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\,(-1)^n\,\cos(nt) - \sum_{n=1}^{\infty}\frac{n}{n^2+1}\,(-1)^n\,\sin(nt)\right)$$

Note that exponential function $f(t) = e^t$ function is *neither even nor odd*, and therefore $a_n \neq 0$ and $b_n \neq 0$. This is a general property of non–even non–odd functions. The sum of first several terms of the Fourier series $f(t)$ is shown in Fig. 5.27 (right). Note that many more terms are needed for good approximation close to the interval's ends.

5.28. Given port function $a\ \Pi\ ({}^{(t-\pi/2)}/_{\pi})$, in $(0, 2\pi)$ interval, (it is not *even*). Thus,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ dt = \frac{1}{\pi}\int_{0}^{\pi} a\ dt = \frac{a}{\cancel{\pi}}\cancel{\pi} = a$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\ dt = \frac{1}{\pi}\int_{0}^{\pi} a\cos(nt)\ dt = \frac{1}{\pi}\frac{a}{n}\ \sin(nt)\Big|_0^{\pi} \\ &= \frac{1}{\pi}\frac{a}{n}\big(\sin(0) - \sin(-n\pi)\big) = \frac{2}{\pi}\frac{a}{n}\ \sin(n\pi) = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\ dt = \frac{1}{\pi}\int_{0}^{\pi} a\sin(nt)\ dt = -\frac{1}{\pi}\frac{a}{n}\ \cos(nt)\Big|_0^{\pi} \\ &= -\frac{1}{\pi}\frac{a}{n}\big(\cos(n\pi) - \cos(0)\big) = -\frac{1}{\pi}\frac{a}{n}\big((-1)^n - 1\big) \end{aligned}$$

Therefore, Fourier series of port function $a\ \Pi\ ({}^{(t-\pi/2)}/_{\pi})$ is

$$\begin{aligned} f(t) &= \frac{a}{2} + \sum_{n=1}^{\infty} 0\ \cos(nt) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{a}{n}\big(1 - (-1)^n\big)\ \sin(nt) \\ &= \frac{a}{2} + \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\big(1 - (-1)^n\big)\ \sin(nt) \\ &= \frac{a}{2} + \frac{2a}{\pi}\ \sin(t) + 0 + \frac{2a}{3\pi}\ \sin(3t) + 0 + \frac{2a}{5\pi}\ \sin(5t) + \cdots \end{aligned}$$

Note that port function $a\ \Pi\ ({}^{(t-\pi/2)}/_{\pi})$ function is *not* even. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal port function, see Fig. 5.28.

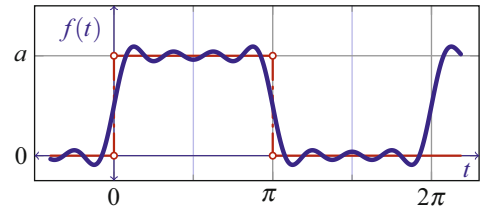


Fig. 5.28 Example P.5.28

5.29. Given port function $a\,\Pi\,(t/\pi)$, in $(-\pi,\pi)$ interval, note that it is an *even* function. Thus, coefficients of Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} a\,dt = \frac{a}{\cancel{\pi}}\cancel{\pi} = a$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} a\cos(nt)\,dt = \frac{1}{\pi}\frac{a}{n}\ \sin(nt)\,\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\frac{a}{n}\big(\sin(n\pi/2) - \sin(-n\pi/2)\big) = \frac{2}{\pi}\frac{a}{n}\ \sin(n\pi/2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} a\sin(nt)\,dt = -\frac{1}{\pi}\frac{a}{n}\ \cos(nt)\,\Big|_{-\pi/2}^{\pi/2}$$

$$= -\frac{1}{\pi}\frac{a}{n}\big(\cos(n\pi/2) - \cos(-n\pi/2)\big) = -\frac{1}{\pi}\frac{a}{n}\big(\cancel{\cos(n\pi/2) - \cos(n\pi/2)}\big) = 0$$

Therefore, Fourier series of port function $a\,\Pi\,(t/\pi)$ is

$$f(t) = \frac{a}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{a}{n}\ \sin(n\pi/2)\cos(nt) + \sum_{n=1}^{\infty} 0\ \sin(nt)$$

$$= \frac{a}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{a}{n}\ \sin(n\pi/2)\cos(nt)$$

$$= \frac{a}{2} + \frac{2a}{\pi}\ \cos(t) + 0 - \frac{2a}{3\pi}\ \cos(3t) + 0 + \frac{2a}{5\pi}\ \cos(5t) + \cdots$$

Note that port function $a\,\Pi\,(t/\pi)$ is *even*, i.e. $f(x) = f(-x)$, and therefore all $b_n = 0$. This is a general property of all even functions. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal port function, see Fig. 5.29.

5.30. Given square function t^2/π^2, in $(-\pi,\pi)$ interval, note that it is an *even* function. Thus, coefficients of Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t^2}{\pi^2}\,dt = \frac{1}{3\pi^3}\,t^3\,\Big|_{-\pi}^{\pi} = \frac{1}{3\pi^3}\big(\pi^3 - (-\pi)^3\big) = \frac{2}{3\cancel{\pi^3}}\cancel{\pi^3} = \frac{2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi^3}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt = \{\text{see Vol. II}\}$$

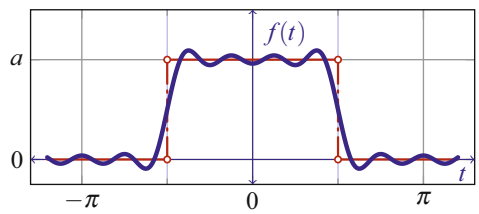
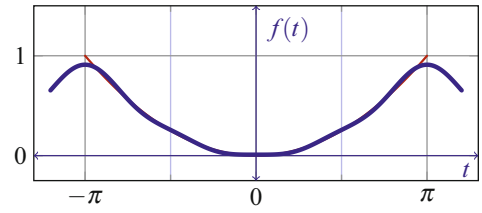


Fig. 5.29 Example P.5.29

Fig. 5.30 Example P.5.30



$$= \frac{1}{\pi^3}\left(\frac{2}{n}\pi^2 \underbrace{\sin(n\pi)}_{=0} - \frac{4}{n^3}\underbrace{\sin(n\pi)}_{=0} + \frac{4}{n^2}\pi \underbrace{\cos(n\pi)}_{(-1)^n}\right) = \frac{1}{\pi^2}\frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin(nt)\,dt = \{\text{see Vol. II}\} = 0$$

Therefore, Fourier series of square function t^2/π^2 is

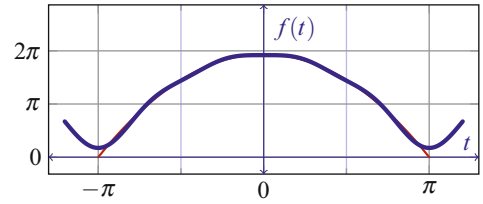
$$\begin{aligned} f(t) &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{1}{\pi^2}\frac{4}{n^2}(-1)^n\cos(nt) + \sum_{n=1}^{\infty} 0\,\sin(nt) \\ &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{1}{\pi^2}\frac{4}{n^2}(-1)^n\cos(nt) \\ &= \frac{1}{3} - \frac{4}{\pi^2}\cos(t) + \frac{4}{\pi^2}\frac{1}{4}\cos(2t) - \frac{4}{\pi^2}\frac{1}{9}\cos(3t) + \frac{4}{\pi^2}\frac{1}{16}\cos(4t) + \cdots \end{aligned}$$

Note that square function t^2/π^2 is *even*, i.e. $f(x) = f(-x)$, and therefore all $b_n = 0$. This is a general property of all even functions. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal square function, see Fig. 5.30.

5.31. Given square function $\pi^2/16\left(\pi^2 - x^2\right)$, in $(-\pi, \pi)$ interval, note that it is an *even* function. Thus, coefficients of Fourier series are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi^2}{16}\left(\pi^2 - t^2\right)dt = \frac{\pi^3}{16}\int_{-\pi}^{\pi} dt - \frac{\pi}{16}\int_{-\pi}^{\pi} t^2\,dt \\ &= \frac{\pi^4}{8} - \frac{\pi^4}{24} = \frac{\pi^4}{12} \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi^2}{16}\left(\pi^2 - t^2\right)\cos(nt)\,dt \\ &= \frac{\pi^3}{16}\underbrace{\int_{-\pi}^{\pi}\cos(nt)\,dt}_{0} - \frac{\pi}{16}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt = \{\text{see P.5.30}\} \\ &= -\frac{\pi^2}{4}\frac{1}{n^2}(-1)^n \end{aligned}$$

Fig. 5.31 Example P.5.31



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi^2}{16}\left(\pi^2 - t^2\right)\sin(nt)\,dt$$

$$= \frac{\pi^3}{16}\int_{-\pi}^{\pi}\sin(nt)\,dt \;(=0) \;-\; \frac{\pi}{16}\int_{-\pi}^{\pi} t^2\sin(nt)\,dt \;(=0) \;= \{\text{see P.5.30}\} = 0$$

Therefore, Fourier series of square function $\pi^2/16\left(\pi^2 - x^2\right)$ is

$$f(t) = \frac{\pi^4}{24} - \sum_{n=1}^{\infty}\frac{\pi^2}{4}\frac{1}{n^2}(-1)^n\cos(nt) + \sum_{n=1}^{\infty} 0\,\sin(nt)$$

$$= \frac{\pi^4}{24} - \sum_{n=1}^{\infty}\frac{\pi^2}{4}\frac{1}{n^2}(-1)^n\cos(nt)$$

$$= \frac{\pi^4}{24} + \frac{\pi^2}{4}\cos(t) - \frac{\pi^2}{4}\frac{1}{4}\cos(2t) + \frac{\pi^2}{4}\frac{1}{9}\cos(3t) - \frac{\pi^2}{4}\frac{1}{16}\cos(4t) + \cdots$$

Note that square function $\pi^2/16\left(\pi^2 - x^2\right)$ is *even*, i.e. $f(x) = f(-x)$, and therefore all $b_n = 0$. This is a general property of all even functions. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal square function, see Fig. 5.31.

5.32. Given triangle function $-2a/\pi\,|t| + a$, in $(-\pi, \pi)$ interval. Thus, coefficients of Fourier series are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(-\frac{2a}{\pi}|t| + a\right)dt = \{\text{even function}\} = \frac{2}{\pi}\int_{0}^{\pi}\left(-\frac{2a}{\pi}t + a\right)dt$$

$$= -\frac{4a}{\pi^2}\int_0^{\pi} t\,dt + \frac{2a}{\pi}\int_0^{\pi} dt = -\frac{4}{\pi^2}\frac{2a}{2}\,t^2\Big|_0^{\pi} + \frac{2a}{\pi}\pi = -\frac{2a}{\pi^2}\pi^2 + 2a = 0$$

Integral of absolute linear function and cosine product should be split in two intervals, as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(-\frac{2a}{\pi}|t| + a\right)\cos(nt)\,dt$$

$$= -\frac{2a}{\pi^2}\int_{-\pi}^{\pi}|t|\cos(nt)\,dt + \frac{a}{\pi}\int_{-\pi}^{\pi}\cos(nt)\,dt$$

$$= -\frac{2a}{\pi^2}\int_{-\pi}^{0}(-t)\cos(nt)\,dt - \frac{2a}{\pi^2}\int_0^{\pi} t\cos(nt)\,dt + \frac{a}{\pi}\int_{-\pi}^{\pi}\cos(nt)\,dt = \{\text{see Vol. II}\}$$

$$= -\frac{2a}{\pi^2}\int_{-\pi}^{0}(-t)\,\cos(nt)\,dt - \frac{2a}{\pi^2}\int_{0}^{\pi} t\,\cos(nt)\,dt + \frac{a}{n\pi}\big(\sin(n\pi) - \sin(-n\pi)\big)$$

$$= -\frac{2a}{\pi^2}\left(-\frac{1}{n}t\,\sin(nt) - \frac{1}{n^2}\cos(nt)\right)\Big|_{-\pi}^{0} - \frac{2a}{\pi^2}\left(\frac{1}{n}t\,\sin(nt) + \frac{1}{n^2}\cos(nt)\right)\Big|_{0}^{\pi}$$

$$= -\frac{2a}{\pi^2}\Big(-\frac{1}{n}0\,\sin(0) - \frac{1}{n^2}\cos(0)\Big) + \frac{2a}{\pi^2}\Big(-\frac{1}{n}(-\pi)\sin(-n\pi) - \frac{1}{n^2}\underbrace{\cos(-n\pi)}_{(-1)^n}\Big)$$

$$- \frac{2a}{\pi^2}\Big(\frac{1}{n}\pi\,\sin(n\pi) + \frac{1}{n^2}\underbrace{\cos(n\pi)}_{(-1)^n}\Big) + \frac{2a}{\pi^2}\Big(\frac{1}{n}0\,\sin(0) + \frac{1}{n^2}\cos(0)\Big)$$

$$= \frac{2a}{n^2\pi^2} - \frac{2a}{n^2\pi^2}(-1)^n - \frac{2a}{n^2\pi^2}(-1)^n + \frac{2a}{n^2\pi^2} = -\frac{4a}{n^2\pi^2}(-1)^n + \frac{4a}{n^2\pi^2}$$

$$= \frac{4a}{n^2\pi^2}\big(1 - (-1)^n\big)$$

Integral of absolute linear function and sine product should be split in two intervals, as

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(-\frac{2a}{\pi}|t| + a\right)\sin(nt)\,dt$$

$$= -\frac{2a}{\pi^2}\int_{-\pi}^{\pi}|t|\,\sin(nt)\,dt + \frac{a}{\pi}\int_{-\pi}^{\pi}\sin(nt)\,dt$$

$$= -\frac{2a}{\pi^2}\left(\int_{-\pi}^{0}(-t)\,\sin(nt)\,dt + \int_{0}^{\pi} t\,\sin(nt)\,dt\right) + \frac{a}{n\pi}\big(-\cos(nt)\big)\Big|_{-\pi}^{\pi}$$

$$= \{\text{see Vol. II}\}$$

$$= -\frac{2a}{\pi^2}\left(\left(\frac{1}{n}t\,\cos(nt) - \frac{1}{n^2}\sin(nt)\right)\Big|_{-\pi}^{0} + \left(-\frac{1}{n}t\,\cos(nt) + \frac{1}{n^2}\sin(nt)\right)\Big|_{0}^{\pi}\right)$$

$$+ \frac{a}{\pi}\frac{1}{n}\underbrace{\big(-\cos(n\pi) - \big(-\cos(-n\pi)\big)\big)}_{=0}$$

$$= -\frac{2a}{\pi^2}\left[\left(\frac{1}{n}0\,\cos(0) - \frac{1}{n^2}\underbrace{\sin(0)}_{=0}\right) - \left(\frac{1}{n}(-\pi)\,\cos(-n\pi) - \frac{1}{n^2}\underbrace{\sin(-n\pi)}_{=0}\right)\right.$$

$$\left. + \left(-\frac{1}{n}\pi\,\cos(n\pi) + \frac{1}{n^2}\underbrace{\sin(n\pi)}_{=0}\right) - \left(-\frac{1}{n}0\,\cos(0) + \frac{1}{n^2}\underbrace{\sin(0)}_{=0}\right)\right]$$

$$= -\frac{2a}{\pi^2}\left(-\frac{1}{n}(-\pi)\,\cos(-n\pi) - \frac{1}{n}\pi\,\cos(n\pi)\right) = -\frac{2a}{n\pi}\big(\cos(n\pi) - \cos(n\pi)\big)$$

$$= 0$$

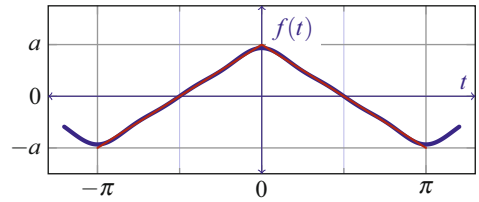


Fig. 5.32 Example P.5.32

Therefore, Fourier series of triangle function $-{}^{2a}/_{\pi}\,|t| + a$ is

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{4a}{n^2\pi^2}\left(1-(-1)^n\right)\cos(nt) \\ &= \frac{8a}{\pi^2}\cos(t) + 0 + \frac{8a}{3^2\pi^2}\cos(3t) + 0 + \frac{8a}{5^2\pi^2}\cos(5t) + \cdots \end{aligned}$$

Note that triangle function $-{}^{2a}/_{\pi}\,|t| + a$ is *even*, i.e. $f(x) = f(-x)$, and therefore all $b_n = 0$. This is a general property of all even functions. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal triangle function, see Fig. 5.32.

5.33. Given triangle function

$$x(t) = \begin{cases} \dfrac{a}{\pi}\,t, & (0 \le t \le \pi) \\ \\ -\dfrac{a}{\pi}\,(t-\pi) + a, & (\pi \le t \le 2\pi) \end{cases}$$

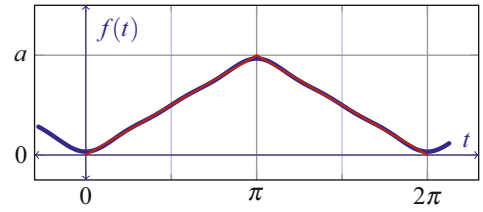
in $(0, 2\pi)$ interval. Thus, coefficients of Fourier series are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi}\frac{a}{\pi}\,t\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(-\frac{a}{\pi}\,(t-\pi)+a\right)dt \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{a}{\pi}\,t\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(-\frac{a}{\pi}\,(t-\pi)+a\right)dt = \frac{a}{\pi^2}\int_0^{\pi} t\,dt - \frac{a}{\pi^2}\int_{\pi}^{2\pi} t\,dt + \frac{2a}{\pi}\int_{\pi}^{2\pi} dt \\ &= \frac{a}{\pi^2}\frac{\pi^2}{2} - \frac{a}{\pi^2}\left(\frac{4\pi^2}{2} - \frac{\pi^2}{2}\right) + \frac{2a}{\pi}\pi = \frac{a}{2} - \frac{3a}{2} + 2a \\ &= a \end{aligned}$$

Integrals a_n of piecewise linear segments may be resolved separately, as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{a}{\pi^2}\int_0^{\pi} t\,\cos(nt)\,dt - \frac{a}{\pi^2}\int_{\pi}^{2\pi} t\,\cos(nt)\,dt + \frac{2a}{\pi}\int_{\pi}^{2\pi}\cos(nt)\,dt \\ &= \{\text{see Vol. II and P.5.32}\} \\ &= -\frac{a}{n^2\pi^2}\left(1-(-1)^n\right) - \frac{a}{n^2\pi^2}\left(1-(-1)^n\right) + 0 \\ &= -\frac{2a}{n^2\pi^2}\left(1-(-1)^n\right) \end{aligned}$$

Fig. 5.33 Example P.5.33



Integrals b_n of piecewise linear segments may also be resolved separately, as

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{a}{\pi^2}\int_0^{\pi} t\,\sin(nt)\,dt - \frac{a}{\pi^2}\int_{\pi}^{2\pi} t\,\sin(nt)\,dt + \frac{2a}{\pi}\int_{\pi}^{2\pi}\sin(nt)\,dt \\ &= -\frac{a}{n\pi}(-1)^n - \frac{a}{n\pi}\big((-1)^n - 2\big) + \frac{2a}{n\pi}\big((-1)^n - 1\big) \\ &= \cancel{-\frac{a}{n\pi}(-1)^n} \cancel{-\frac{a}{n\pi}(-1)^n} + \cancel{\frac{2a}{n\pi}} + \cancel{\frac{2a}{n\pi}(-1)^n} \cancel{-\frac{2a}{n\pi}} \\ &= 0 \end{aligned}$$

Therefore, Fourier series of this triangle function is

$$\begin{aligned} f(t) &= \frac{a}{2} - \sum_{n=1}^{\infty}\frac{2a}{n^2\pi^2}\big(1-(-1)^n\big)\cos(nt) - \sum_{n=1}^{\infty} 0\,\sin(nt) \\ &= \frac{a}{2} - \frac{4a}{\pi^2}\cos(t) - \frac{4a}{3^2\pi^2}\cos(3t) - \frac{4a}{5^2\pi^2}\cos(5t) + \cdots \end{aligned}$$

The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal triangle function, see Fig. 5.33.

Discrete-Time Fourier Transform

6

Fourier sum is defined as

$$\mathscr{F}(x_n) = X_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x_n \, e^{-j\frac{2\pi\, k}{N}\, n}$$

$$\mathscr{F}^{-1}(X[k]) = x[n] = \frac{1}{T} \sum_{-\infty}^{\infty} X[k] \, \exp\left(j\frac{2\pi k}{N} n\right)$$

Fourier series is defined as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{j\,\omega_0\, t}\,, \quad \omega_0 = \frac{2\pi}{T}$$

$$C_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-j\,\omega_0\, t}\, dt$$

Problems

6.1 Simple Seqences

Given functions $x[n]$ in P.6.1 to P.6.6 (assume $(|\alpha| < 1, \tau > 0)$, derive Fourier transformations $X(\omega)$.

6.1. $x[n] = \{1, 2, 3, 4\}$

6.2. $x[n] = \{2, 3, -1, 1\}$

6.3. $x[n] = \delta[n], \quad N = 16$

6.4. $x[n] = \delta[n - n_0]$

6.5. $x[n] = 1, \quad N = 16$

6.6. $x[n] = \alpha^n \, u[n - 1]$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_6

6.2 Fourier Series

Given functions $x(t)$ in P.6.7 to P.6.12, derive their Fourier series.

6.7. Derive discrete Fourier transformations $X[k]$ of the following functions $x(t)$.
$x(t) = \sin(1\,\text{Hz})$, sampling frequency $f_s = 8\,\text{Hz}$.

6.8. Given $f(t) = 1 - (t/\pi)$, derive its Fourier series if $t \in [-\pi, \pi]$.

6.9. Given square waveform in Fig. 6.1, derive Fourier series assuming $a = T/4$.

6.10. Given square waveform in Fig. 6.1, derive Fourier series assuming $a = T/8$.

6.11. Given square waveform in Fig. 6.2, derive its Fourier series.

6.12. Given $x(t)$ waveform in Fig. 6.3 derive its Fourier series.

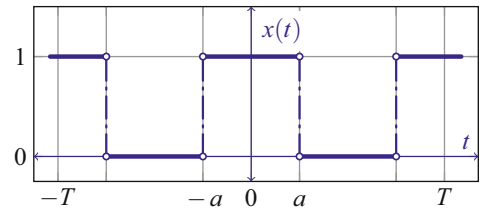


Fig. 6.1 Example P.6.9

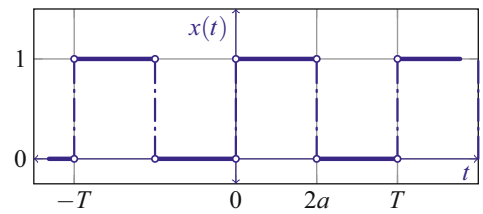


Fig. 6.2 Example P.6.11

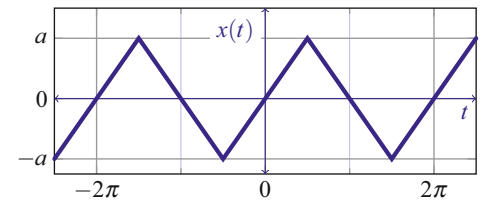


Fig. 6.3 Example P.6.12

6.3 Inverse Discrete Time FT

Given discrete sequences $X[k]$ in P.6.13 to P.6.14, derive their inverse Fourier transforms $x[n]$.

6.13. $X[k] = \{10, -2 + 2j, -2, -2 - 2j\}$

6.14. $X[k] = \{5, 3 - j2, -3, 3 + j2\}$

6.4 Fast Fourier Transform

Resolve questions in P.6.15 to P.6.21.

6.15. Assuming the audio spectrum signal, [20 Hz, 20 kHz], estimate the time needed to calculate discrete Fourier transformation (DFF) by hands, if the desired resolution is 1 Hz and it takes 1min to perform one calculation (extremely fast, but for the sake of argument). Then, compare with the time needed to do the same calculations by using fast Fourier transformation (FFT).

6.16. Given a general discrete series $x_n = x[n]$, briefly develop the FFT equations and explain the main idea behind it.

6.17. Given a general discrete series $x_n = x[n]$ and $N = 2$, apply FFT algorithm.

6.18. Given a general discrete series $x_n = x[n]$ and $N = 4$, apply FFT algorithm.

6.19. By using FFT graph algorithm in P.6.18, calculate Fourier transformation of $x[n] = \{1, 2, 3, 4\}$ (it is already solved using DFT in P.6.1).

6.20. Further to discussion in P.6.16, deduce FFT graph diagram if number of transformation points is $N = 8$.

6.21. By using FFT graph algorithm in P.6.20, calculate $\mathscr{F}(x_n)$ if $x_n = [1, 1, -1, -1, 1, 1, -1, -1]$.

Answers

6.1 Simple Seqences

6.1. Given $x[n] = \{1, 2, 3, 4\}$, DT FT in $N = 4$ points is derived by definition

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi k}{4}n\right)$$

In discrete sequence interval, both $n, k \in (0, 3)$, thus for each case k it follows that,

$$\underline{k=0}: \quad X[0] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi\ 0}{4}n\right) = \sum_{n=0}^{3} x[n]\, e^{0} \to 1$$
$$= x[0] + x[1] + x[2] + x[3] = 1 + 2 + 3 + 4$$
$$= 10$$

$$\underline{k=1}: \quad X[1] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi\ 1}{4}n\right) = \sum_{n=0}^{3} x[n] \underbrace{\exp\left(-j\frac{\pi}{2}n\right)}_{(-j)^n}$$
$$\{\text{if needed, review calculations of complex number powers}\}$$
$$= x[0](-j)^0 + x[1](-j)^1 + x[2](-j)^2 + x[3](-j)^3$$
$$= 1(1) + 2(-j) + 3(-1) + 4(j) = 1 - 2j - 3 + 4j$$
$$= -2 + 2j$$

$$\underline{k=2}: \quad X[2] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi\ 2}{4}n\right) = \sum_{n=0}^{3} x[n] \underbrace{\exp(-j\pi n)}_{(-1)^n}$$
$$= x[0](-1)^0 + x[1](-1)^1 + x[2](-1)^2 + x[3](-1)^3$$
$$= 1(1) + 2(-1) + 3(1) + 4(-1) = 1 - 2 + 3 - 4$$
$$= -2$$

$$\underline{k=3}: \quad X[3] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi\ 3}{4}n\right) = \sum_{n=0}^{3} x[n] \underbrace{\exp\left(-j\frac{3\pi}{2}n\right)}_{(j)^n}$$
$$= x[0](j)^0 + x[1](j)^1 + x[2](j)^2 + x[3](j)^3$$
$$= 1(1) + 2(j) + 3(-1) + 4(-j) = 1 + 2j - 3 - 4j$$
$$= -2 - 2j = x[1]^*$$

Recall that as $\Theta = -\pi \equiv \pi$ and $\Theta = -(3\pi/4) \equiv (5\pi/4)$, then the complete solution may be written as,

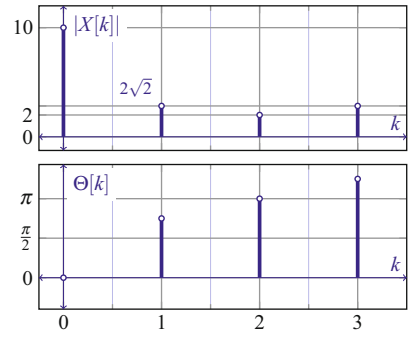


Fig. 6.4 Example P.6.1

$$X[k] = \{10, -2+2j, -2, -2-2j\}$$
$$|X[k]| = \left\{10, 2\sqrt{2}, 2, 2\sqrt{2}\right\}$$
$$\Theta[k] = \left\{0, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}\right\}$$

where $|X[k]|$ and phase $\Theta[k]$ diagrams are in Fig. 6.4

6.2. Given $x[n] = \{2, 3, -1, 1\}$, DT FT in $N = 4$ points is derived by definition

$$X[k] = \sum_{n=0}^{N-1} x[n]\, \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=0}^{3} x[n]\, \exp\left(-j\frac{2\pi k}{4}n\right)$$

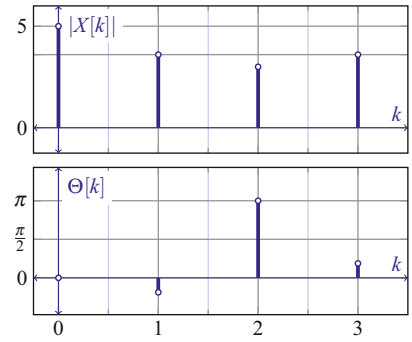
As both $n, k \in (0, 3)$ interval, thus for each case k it follows that,

$$\underline{k=0}\ :\quad X[0] = \sum_{n=0}^{3} x[n]\, \exp\left(-j\frac{\pi\ 0}{2}n\right) = \sum_{n=0}^{3} x[n] \exp(0) = 1 \sum_{n=0}^{3} x[n]$$
$$= 2+3-1+1 = 5$$

$$\underline{k=1}\ :\quad X[1] = \sum_{n=0}^{3} x[n]\, \exp\left(-j\frac{\pi\ 1}{2}n\right)$$
$$= 2\exp\left(-j\frac{0\pi}{2}\right) + 3\exp\left(-j\frac{1\pi}{2}\right) - 1\exp\left(-j\frac{2\pi}{2}\right) + 1\exp\left(-j\frac{3\pi}{2}\right)$$
$$= 2 + 3(-j) - 1(-1) + 1(j)$$
$$= 3 - 2j$$

$$\underline{k=2}\ :\quad X[2] = \sum_{n=0}^{3} x[n]\, \exp\left(-j\frac{\pi\ 2}{2}n\right)$$
$$= 2\exp\left(-j\left(0\pi\right)\right) + 3\exp\left(-j\left(1\pi\right)\right) - 1\exp\left(-j\left(2\pi\right)\right) + 1\exp\left(-j\left(3\pi\right)\right)$$

Fig. 6.5 Example P.6.2



$$= 2 + 3(-1) - 1(1) + 1(-1)$$
$$= -3$$

$$\underline{k = 3} \;:\; X[3] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{\pi}{2}\frac{3}{n}\right)$$
$$= 2\exp\left(-j\frac{\pi}{2}\frac{3}{0}\right) + 3\exp\left(-j\frac{\pi}{2}\frac{3}{1}\right) - 1\exp\left(-j\frac{\pi}{2}\frac{3}{2}\right) + 1\exp\left(-j\frac{\pi}{2}\frac{3}{3}\right)$$
$$= 2 + 3(j) - 1(-1) + 1(-j)$$
$$= 3 + 2j$$

Recall that as $\pi \equiv 180°$, then the complete solution may be written as,

$$X[k] = \{5, 3 - 2j, -3, 3 + 2j\}$$
$$|X[k]| = \left\{5, \sqrt{13}, 3, \sqrt{13}\right\}$$
$$\Theta[k] = \{0, -33.7°, \pi, 33.7°\}$$

where $|X[k]|$ and phase $\Theta[k]$ diagrams are in Fig. 6.5

6.3. Given Delta function $x[n] = \delta[n]$ in $N = 16$ points, that is to say $x[0] = 1$ otherwise $x[n] = 0, n \neq 0$, or

$$x[n] = \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

by definition it follows that,

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=0}^{15} x[n] \exp\left(-j\frac{2\pi k}{\cancel{16}\ 8}n\right)$$

Calculations for each k are as follows.

$$
\begin{aligned}
\underline{k=0}\ :\quad X[0] &= \sum_{n=0}^{15} x[n]\ \exp\left(-j\frac{\pi\ 0}{8}n\right) = \sum_{n=0}^{15} x[n] \exp(0) \to 1 \\
&= x[0]+x[1]+x[2]+\cdots+x[15] = 1+0+0+\cdots+0 \\
&= 1 \\
\underline{k=1}\ :\quad X[1] &= \sum_{n=0}^{15} x[n]\ \exp\left(-j\frac{\pi\ 1}{8}n\right) = \sum_{n=0}^{15} x[n]\ \exp\left(-j\frac{\pi}{8}n\right) \\
&= x[0]\ \exp\left(-j\frac{\pi}{8}0\right)+0+0+\cdots+0 \\
&= 1 \\
\underline{k=2}\ :\quad X[2] &= \sum_{n=0}^{15} x[n]\ \exp\left(-j\frac{\pi\ 2}{8}n\right) = \sum_{n=0}^{15} x[n]\ \exp\left(-j\frac{2\pi}{8}n\right) \\
&= x[0]\ \exp\left(-j\frac{2\pi}{8}0\right)+0+0+\cdots+0 \\
&= 1 \\
&\vdots \\
&\text{etc.}
\end{aligned}
$$

In conclusion, given $x[k]=\delta[k]$, then $X[k]=1,\ \forall k$, which is a constant function that is well known to be dual function to Dirac impulse.

6.4. **Method 1:** DT FT of delayed impulse $x[n]=\delta[n-n_0]=\{\cdots 0\ 0\cdots 1\cdots 0\ 0\cdots\}$ in N points, where $n=n_0$ is indice of the only non–zero sample may be derived by definition as

$$
\begin{aligned}
X[k] &= \sum_{n=-\infty}^{\infty} x[n]\ \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=-\infty}^{\infty} \delta[n-n_0]\ \exp\left(-j\frac{2\pi k}{N}n\right) \\
&= \cdots+0+0+\underbrace{\delta[n_0-n_0]\ \exp\left(-j\frac{2\pi k}{N}n_0\right)}_{n=n_0}+0+0+\cdots \\
&= 1\ \exp\left(-j\frac{2\pi k}{N}n_0\right)
\end{aligned}
$$

Therefore, $|X[k]|=1, \forall k$ and its argument is $(2\pi k\, n_0/N)$, which illustrates the property of delayed functions in general where the delay transforms into the equivalent argument (i.e. angle or phase).

Method 2: knowing time domain delay property of FT and that $\delta[n] \overset{\text{FT}}{\to} 1$, then it follows directly that

$$
\mathscr{F}(\delta[n-n_0]) = \mathscr{F}(\delta[n])\ \exp\left(-j\frac{2\pi k}{N}n_0\right) \quad\Rightarrow\quad X[k]=1\ \exp\left(-j\frac{2\pi k}{N}n_0\right)
$$

6.5. Given constant function $x[n] = 1$ in $N = 16$ points may be transformed by definition as

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=0}^{15} 1 \exp\left(-j\frac{2\pi k}{16\ 8}n\right)$$

As both $n, k \in (0, 15)$ interval, thus for each case k it follows that,

$$\begin{aligned}
\underline{k=0}: \quad X[0] &= \sum_{n=0}^{15} \exp\left(-j\frac{\pi\ 0}{8}n\right) = \sum_{n=0}^{15} \exp(0) \to 1 = \sum_{n=0}^{15} 1_n \\
&= 16 \\
\underline{k=1}: \quad X[1] &= \sum_{n=0}^{15} \exp\left(-j\frac{\pi\ 1}{8}n\right) = \sum_{n=0}^{15} \exp\left(-j\frac{\pi}{8}n\right) \\
&= \exp\left(-j\frac{\pi}{8}\,0\right) + \exp\left(-j\frac{\pi}{8}\,1\right) + \cdots + \exp\left(-j\frac{\pi}{8}\,15\right)
\end{aligned}$$

The question is: how to add these sixteen exponential terms? Recall that modules of all complex exponential terms equal one (the unit circle radius). See Vol. I to recall the exponential form of complex numbers. Finally, sampled complex terms are symmetrically and periodically distributed over one period, $T = 2\pi$ i.e. one circle circumference. Given $N = 16$ points, each term in the sum is a complex number that is shifted along the circle at the multiples of $2\pi/N = 2\pi/16 = \pi/8$, see Fig. 6.6. Therefore, the summation may be done by considering symmetrical pairs of numbers.

For example, see Fig. 6.6, $x[6]$ and $x[14]$ are at the opposite sides and by consequence their real parts $\pm$Re have opposite signs, as well as their imaginary parts $\pm$Im, therefore their sum equals zero as

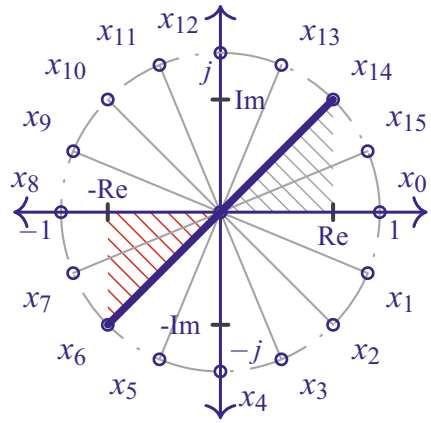
$$x[14] + x[6] = \Re(x[6]) + j\Im(x[14]) - \Re(x[6]) - \Im(x[6]) = 0$$

Or, more formally, this conclusion may be confirmed as

$$\begin{aligned}
x[6] + x[14] &= \exp\left(-j\frac{\pi}{8}\,6\right) + \exp\left(-j\frac{\pi}{8}\,14\right) \\
&= \exp\left(-j\frac{2\,\pi}{4\ 8}\,3\right) + \exp\left(-j\frac{2\,\pi}{4\ 8}\,7\right) \\
&= \exp\left(-j\frac{3\pi}{4}\right) + \exp\left(j\frac{\pi}{4}\right) \\
&= \cos\left(-\frac{3\pi}{4}\right) + j\sin\left(-\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) \\
&= -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \\
&= 0
\end{aligned}$$

Similarly, it may be shown that all the remaining sums of the opposite complex pairs also equal zero, i.e.

Fig. 6.6 Example P.6.5



$$x[0] + x[8] = 0 \qquad x[4] + x[12] = 0$$
$$x[1] + x[9] = 0 \qquad x[5] + x[13] = 0$$
$$x[2] + x[10] = 0 \qquad x[6] + x[14] = 0$$
$$x[3] + x[11] = 0 \qquad x[7] + x[15] = 0$$

In conclusion, $X[1] = 0$
k = 2, 3, ... 15 :

After following the same reasoning, it may be shown that for all the remaining values of $k = 2, 3, \ldots 15$ it is true that $X[k]$, therefore

$$X[0] = 16$$
$$X[1] = \exp\left(-j\frac{\pi}{8}0\right) + \exp\left(-j\frac{\pi}{8}1\right) + \cdots + \exp\left(-j\frac{\pi}{8}15\right) = 0$$
$$X[2] = X[3] = \cdots = X[15] = 0$$

In summary, given $N = 16$

$$\mathscr{F}(x[n] = 1) = 16\,\delta[n]$$

and $X[k]$ being real function, the argument $\theta[k] = 0$.

6.6. Given infinitely long exponential series $x[n] = \alpha^n\, u[n-1]$, where $|\alpha| < 1$, its DT FT may be derived by definition as

$$X[k] = \sum_{n=-\infty}^{\infty} x[n] \exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=-\infty}^{\infty} \alpha^n\, u[n-1] \exp\left(-j\frac{2\pi k}{N}n\right)$$

$$\left\{\text{delayed step function } u[n-1] = 0,\ \forall n < 1 \ \ \therefore\ \ u[n-1] = 1,\ \forall n \geq 1\right\}$$

$$= \sum_{n=1}^{\infty} \alpha^n \, (1) \, \exp\left(-j\frac{2\pi k}{N} n\right) = \sum_{n=1}^{\infty} \left[\alpha \, \exp\left(-j\frac{2\pi k}{N}\right)\right]^n$$

$$\left\{x^a \, y^a = (xy)^a; \quad \text{as well as} \quad x^{ab} = \left(x^a\right)^b\right\}$$

$$\left\{\sum_{k=m}^{N} a^k = \frac{a^{N+1} - a^m}{a-1}, \quad (|a| < 1)\right\}$$

$$= \frac{\left[\alpha \, \exp\left(-j\frac{2\pi k}{N}\right)\right]^{\infty+1} - \left[\alpha \, \exp\left(-j\frac{2\pi k}{N}\right)\right]^1}{\alpha \, \exp\left(-j\frac{2\pi k}{N}\right) - 1}$$

$$\left\{\left|\exp\left(-j\frac{2\pi k}{N}\right)\right| = 1, \quad \alpha < 1 \;\; \therefore \;\; \alpha \, \exp\left(-j\frac{2\pi k}{N}\right) < 1 \;\; \text{see Vol. I}\right\}$$

$$= \frac{\alpha \, \exp\left(-j\frac{2\pi k}{N}\right)}{1 - \alpha \, \exp\left(-j\frac{2\pi k}{N}\right)}$$

6.2 Fourier Series

6.7. Given a CT sine function $x(t) = \sin(1\,\text{Hz})$, before performing DT FT it is necessary to convert $x(t) \to x[n]$, which is done in the following steps.

1. Sampling frequency normalization : good practical choice for N is to use the powers of two, here $N = 2^3 = 8$ points. Given the sampling frequency $f_s = 8\,\text{Hz}$ and the sine function's period $T(1\,\text{Hz}) = 1\text{s}$, it follows that

$$f = 1\,\text{Hz} \;\; \Rightarrow \;\; T = \frac{1}{f} = 1\text{s} \Leftrightarrow 2\pi$$

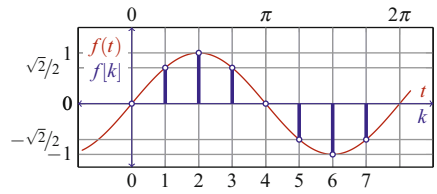
$$f_s = 8\,\text{Hz} \Rightarrow N = 8$$

$$\therefore$$

$$\frac{1}{8}\text{s} \equiv \frac{2\pi}{8} = \frac{\pi}{4}$$

   that is to say, each sample is found at the multiples of $\pi/4$, see Fig. 6.7. Note that first sample $x[0] = 0$, as is the first sample of the next period.
2. Calculations of each sample : $N = 8$ samples are evenly distributed over one period, i.e. full circle, therefore separated by $\pi/4$ angle. Each sample is then calculated as,

Fig. 6.7 Example P.6.7



$$x[0] = (1)\sin\left(0\frac{\pi}{4}\right) = 0 \qquad x[4] = (1)\sin\left(4\frac{\pi}{4}\right) = 0$$

$$x[1] = (1)\sin\left(1\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad x[5] = (1)\sin\left(5\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$x[2] = (1)\sin\left(2\frac{\pi}{4}\right) = 1 \qquad x[6] = (1)\sin\left(6\frac{\pi}{4}\right) = -1$$

$$x[3] = (1)\sin\left(3\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad x[7] = (1)\sin\left(7\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Therefore,

$$x[n] = \left\{0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}\right\}$$

Note that simple sum of samples is

$$\sum_{n=1}^{7} x[n] = 0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1 - \frac{\sqrt{2}}{2} = 0$$

3. $x(t) \overset{\text{FT}}{\rightarrow} X[k]$: given $N = 8$ by definition it follows that

$$X[k] = \sum_{n=0}^{N-1} x[n]\,\exp\left(-j\frac{2\pi k}{N}n\right) = \sum_{n=0}^{7} x[n]\,\exp\left(-j\frac{2\pi k}{8}n\right)$$

so that calculations for each case k are as follows.

$$k = 0 \;:\; X[0] = \sum_{n=0}^{7} x[n]\,\exp\left(-j\frac{\pi\,0}{4}n\right) = \sum_{n=0}^{7} x[n]\,e^{0} = \sum_{n=0}^{7} x[n] = 0$$

$$k = 1 \;:\; X[1] = \sum_{n=0}^{7} x[n]\,\exp\left(-j\frac{\pi\,1}{4}n\right)$$

$$= (0)\exp(0) + \frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{4}1\right) + (1)\exp\left(-j\frac{\pi}{4}2\right)$$

$$+ \frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{4}3\right) + (0)\exp\left(-j\frac{\pi}{4}4\right) - \frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{4}5\right)$$

$$- (1)\exp\left(-j\frac{\pi}{4}6\right) - \frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{4}7\right)$$

$$\{\cos(-x)=\cos(x)\quad \sin(-x)=-\sin(x)\}$$

$$= 0+\frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4}\right)-j\sin\left(\frac{\pi}{4}\right)\right]+(1)\left[\cos\left(\frac{2\pi}{4}\right)-j\sin\left(\frac{2\pi}{4}\right)\right]+\cdots$$

$$= 0+(0.5-j0.5)+(-j)+(-0.5-j0.5)$$

$$+0+(0.5-j0.5)+(-j)+(-0.5-j0.5)$$

$$=-4j$$

$$\underline{k=2}\ :\quad X[2]=\sum_{n=0}^{7}x[n]\exp\left(-j\frac{\pi}{4}\frac{2}{2}n\right)$$

$$= (0)\exp(0)+\frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{2}\right)+(1)\exp\left(-j\frac{\pi}{2}2\right)$$

$$+\frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{2}3\right)+(0)\exp\left(-j\frac{\pi}{2}4\right)-\frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{2}5\right)$$

$$-(1)\exp\left(-j\frac{\pi}{2}6\right)-\frac{\sqrt{2}}{2}\exp\left(-j\frac{\pi}{2}7\right)$$

$$=\frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{2}\right)-j\sin\left(\frac{\pi}{2}\right)\right]+(1)\left[\cos\left(\frac{2\pi}{2}\right)-j\sin\left(\frac{2\pi}{2}\right)\right]+\cdots$$

$$=-j\frac{\sqrt{2}}{2}-1+j\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2}+1-j\frac{\sqrt{2}}{2}$$

$$=0$$

Similarly, $X[2]=X[3]=X[4]=X[5]=X[6]=0$. However,

$$\underline{k=7}\ :\quad X[7]=\sum_{n=0}^{7}x[n]\exp\left(-j\frac{\pi}{4}\frac{7}{}n\right)$$

$$= (0)\exp(0)+\frac{\sqrt{2}}{2}\exp\left(-j\frac{7\pi}{4}1\right)+(1)\exp\left(-j\frac{7\pi}{4}2\right)$$

$$+\frac{\sqrt{2}}{2}\exp\left(-j\frac{7\pi}{4}3\right)+(0)\exp\left(-j\frac{7\pi}{4}4\right)-\frac{\sqrt{2}}{2}\exp\left(-j\frac{7\pi}{4}5\right)$$

$$-(1)\exp\left(-j\frac{7\pi}{4}6\right)-\frac{\sqrt{2}}{2}\exp\left(-j\frac{7\pi}{4}7\right)$$

$$\{\cos(-x)=\cos(x)\quad \sin(-x)=-\sin(x)\}$$

$$\left\{\text{use unit circle to convert angles, for example } -\frac{7\pi}{4}\equiv\frac{\pi}{4}\text{, etc.}\right\}$$

$$= 0+\frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4}\right)+j\sin\left(\frac{\pi}{4}\right)\right]+(1)\left[\cos\left(\frac{\pi}{2}\right)+j\sin\left(\frac{\pi}{2}\right)\right]+\cdots$$

$$
\begin{aligned}
&= 0 + (\cancel{0.5} + j0.5) + (+j) + (\cancel{-0.5} + j0.5) \\
&\quad + 0 + (\cancel{0.5} + j0.5) + (+j) + (\cancel{-0.5} + j0.5) \\
&= 4j
\end{aligned}
$$

In summary, given $N = 8$, $f_s = 8\,\text{Hz}$, and $x(t) = \sin(1\,\text{Hz})$, DT FT is

$$
\begin{aligned}
X[k] &= \{0, -4j, 0, 0, 0, 0, 0, 4j\} \\
|X[k]| &= \{0, 4, 0, 0, 0, 0, 0, 4\}
\end{aligned}
$$

Note that Fourier transformation result $X[k]$ is "bilateral", see Fig. 6.8 (top), there are two non–zero impulses in $|X[k]|$ sequence. At this stage of calculations, FT result is relative strictly to an *abstract* variable k.

4. In order to express the amplitude $|X[k]|$ relative to the physical units of frequency, i.e. [Hz], resolution of each frequency "bin" is calculated as

$$
\frac{f_s}{N} = \frac{8\,\text{Hz}}{8} = 1\,\text{Hz/bin}
$$

In this case, because N and sampling frequency f_s are numerically equal it happens that k maps directly into Hz. That is to say, by selecting different either N or f_s the resolution of Fourier transformation may be set arbitrarily. In addition, impulses found above half of sampling frequency, known as Nyquist frequency, are folded over to create "unilateral" view of Fourier transformation in 0 to $f_s/2$ range (as commonly shown by spectrum analyzers or simulated graphs), see Fig. 6.8 (bottom).
Consequently, in unilateral view amplitudes at each frequency are calculated as the sum of samples already in place and the ones "folded over".
For example amplitude of sample at $x(1\,\text{Hz})$ is increased by amplitude of $x(7\,\text{Hz})$ that after folding over falls on top, thus $|H[1\,\text{Hz}]| = 4 + 4 = 8$. This amplitude is the consequence of $N = 8$, thus in order to show correct amplitude, i.e $|x(t)| = 1$ the unilateral graph must be renormalized by dividing the amplitudes by N, see Fig. 6.9.

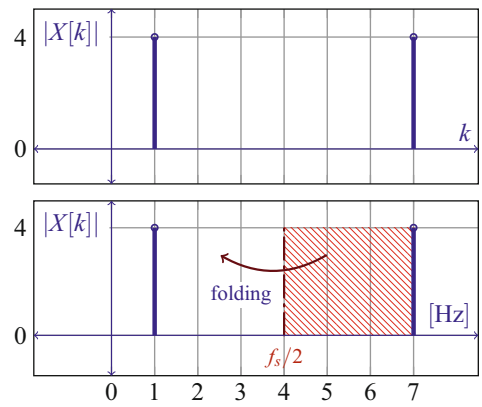


Fig. 6.8 Example P.6.7

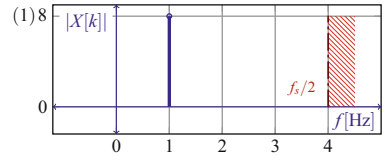


Fig. 6.9 Example P.6.7

6.8. Given a linear function $f(t) = 1 - {}^{t}/_{\pi}$ its Fourier series within the periodic interval $x \in [-\pi, \pi]$ is calculated by definition as,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\; dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\;\cos(n\,t)\; dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\;\sin(n\,t)\; dt$$

Therefore, given period $T = 2\pi$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{t}{\pi}\right)\; dt = \frac{1}{\pi}\left[\, t\,\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\, t^2\,\Big|_{-\pi}^{\pi}\right] = \frac{1}{\pi}\left[2\pi - \frac{1}{2\pi}\left(\cancel{\pi^2} - \cancel{\pi^2}\right)\right]$$

$$= 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{t}{\pi}\right)\;\cos(n\,t)\; dt = \{\text{see Vol. II}\}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(n\,t)\; dt - \frac{1}{\pi^2}\left[\underbrace{\int_{-\pi}^{\pi} x\;\cos(n\,t)\; dt}_{0}\right]$$

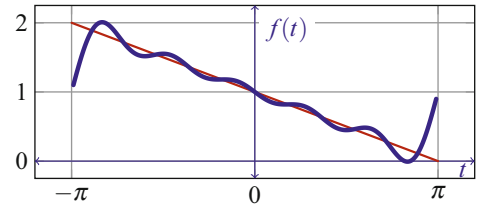
$$= \frac{1}{n\pi}\;\sin(nt)\,\Big|_{-\pi}^{\pi} = \frac{1}{n\pi}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(-n\pi)}_{0}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{t}{\pi}\right)\;\sin(n\,t)\; dt = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin(n\,t)\; dt}_{0} - \frac{1}{\pi^2}\int_{-\pi}^{\pi} t\;\sin(n\,t)\; dt = \{\text{see Vol. II}\}$$

$$= -\frac{1}{\pi^2}\frac{1}{n}\left[\, -t\cos(nt) + \underbrace{\sin(nt)}_{\sin(n\pi)=0}\,\right]_{-\pi}^{\pi} = \frac{1}{n\pi^2}\left[\,\pi\cos(n\pi) + \pi\cos(n\pi)\,\right]$$

Fig. 6.10 Example P.6.8



$$= \frac{1}{n\pi} 2 \underbrace{\cos(n\pi)}_{(-1)^n}$$

$$= (-1)^n \frac{2}{n\pi}$$

Therefore,

$$f(t) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi} \sin(nt) = 1 - \frac{2}{\pi}\sin(t) + \frac{2}{2\pi}\sin(2t) - \frac{2}{3\pi}\sin(3t) + \cdots$$

This series is superimposed to $f(t)$, see Fig. 6.10, which illustrates the series approximation. To achieve "perfect" approximation, there should be infinite number of terms. The largest discrepancy between the continuous function $f(t)$ and its approximation is at the ends of given interval. Over wider interval Fourier series is a periodic function.

6.9. Note that this square pulse waveform is even as $f(-t) = f(t)$ (i.e. symmetric around $t = 0$).
Exponential form of Fourier series, by definition, is

$$x(t) = \sum_{k=-\infty}^{\infty} C_n \exp(j n\omega_0 t); \quad \text{where,} \quad \omega_0 = \frac{2\pi}{T} \quad \text{and}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) \exp(-j n\omega_0 t)\, dt$$

Case $n = 0$: note that $x(t) = 1, \quad t \in [-a, a]$, thus

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) \exp(-j\,0\,\omega_0 t)\, dt = \frac{1}{T}\int_{-a}^{a} (1)\, dt = \frac{2a}{T}$$

Case $n \geq 1$:

$$C_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \exp(-j n\omega_0 t)\, dt = \frac{2}{T}\int_{-a}^{a} (1) \exp(-j n\omega_0 t)\, dt$$

$$= \frac{2}{-jn\omega_0}\frac{1}{T}\left[\exp(-j n\omega_0 a) - \exp(j n\omega_0 a)\right]$$

$$= \frac{2\cancel{T}}{n\pi}\frac{1}{\cancel{T}} \frac{\exp(j\,n\omega_0 a) - \exp(-j\,n\omega_0 a)}{2j}$$
$$= \frac{2}{n\pi}\sin\left(\frac{2n\pi}{T}a\right)$$

Then, given specific high/low ratio,
$a = T/4$: therefore,

$$C_0 = \frac{2a}{T} = \frac{2\cancel{T}}{\cancel{T}\cancel{4}\,2} = \frac{1}{2}$$
$$C_n = \frac{2}{n\pi}\sin\left(\frac{2n\pi}{T}a\right) = \frac{2}{n\pi}\sin\left(\frac{\cancel{2}n\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{4}\,2}\right) = \frac{\sin(n\pi/2)}{n\pi/2} = \text{sinc}\,(n\pi/2)$$
$$\therefore$$
$$|C_n| = |\text{sinc}\,(n\pi/2)\,|$$

A few calculated samples in the series are, see Fig. 6.11

$$\frac{\sin\left(n\frac{\pi}{2}\right)}{n\frac{\pi}{2}} = \begin{cases} C_1 & = {}^{2}/_{\pi}\sin({}^{\pi}/_{2}) = \dfrac{2}{\pi} \\ C_2 & = {}^{2}/_{2\pi}\sin({}^{2\pi}/_{2}) = 0 \\ C_3 & = {}^{2}/_{3\pi}\sin({}^{3\pi}/_{2}) = -\dfrac{2}{3\pi} \\ C_4 & = {}^{2}/_{4\pi}\sin({}^{4\pi}/_{2}) = 0 \\ C_5 & = {}^{2}/_{5\pi}\sin({}^{5\pi}/_{2}) = \dfrac{2}{5\pi} \\ & \cdots \end{cases}$$

That is to say, function $x(t)$ can be synthesized as the sum of infinite sine series, where all even terms equal zero. Being a complex number $C_n = a_n + jb_n$, recall that $|C_n| = \sqrt{a_n^2 + b_n^2}$, thus

$$x(t) = \sum_{k=-\infty}^{\infty} C_n\ \exp(j\,n\omega_0 t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$
$$\left\{\text{for even functions } b_n = 0 \ \ \therefore\ \ |a_n| = |C_n|\right\}$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\,\omega_0\,t) = \left\{\omega_0 = \frac{2\pi}{T},\ \ a = \frac{T}{4}\right\}$$
$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos\left(\frac{2\pi}{T}\,t\right) - \frac{1}{3}\cos\left(3\times\frac{2\pi}{T}\,t\right) + \frac{1}{5}\cos\left(5\times\frac{2\pi}{T}\,t\right) - \cdots\right]$$

As a comparison, this series is overlaid on $x(t)$, see Fig. 6.12 where only the first four terms are used, and spectrum is illustrated in Fig. 6.11. Note that all even terms equal zero.

6.10. Note that this square pulse waveform is even as $f(-t) = f(t)$ (i.e. symmetric around $t = 0$).

Fig. 6.11 Example P.6.9

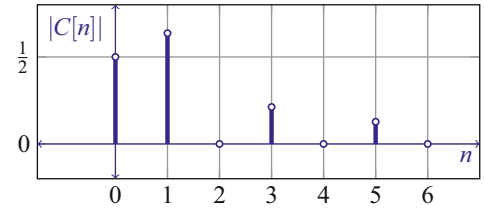
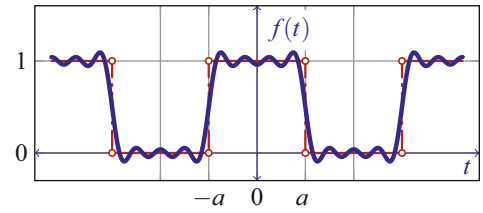


Fig. 6.12 Example P.6.9



See A.6.9, where after setting $a = T/8$ it follows that

$$C_0 = \frac{2a}{T} = \frac{2\cancel{T}}{\cancel{T}\cancel{8}\,4} = \frac{1}{4}$$

$$C_n = \frac{2}{n\pi}\sin\left(\frac{2n\pi}{T}a\right) = \frac{2}{2}\frac{2}{n\pi}\sin\left(\frac{\cancel{2}n\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{8}\,4}\right) = \frac{1}{2}\frac{\sin(n\pi/4)}{n\pi/4} = \frac{1}{2}\,\mathrm{sinc}\,(n\pi/4)$$

$$\therefore$$

$$|C_n| = \frac{1}{2}\,|\mathrm{sinc}\,(n\pi/4)\,|$$

A few calculated samples in the series are, see Fig. 6.13

$$\frac{1}{\cancel{2}}\frac{\sin\left(n\frac{\pi}{4}\right)}{\frac{n\,\pi}{\cancel{4}\,2}} = \begin{cases} C_1 &= {}^{2}/_{\pi}\sin({}^{\pi}/_{4}) = \dfrac{\sqrt{2}}{\pi} \\ C_2 &= {}^{2}/_{2\pi}\sin({}^{2\pi}/_{4}) = \dfrac{1}{\pi} \\ C_3 &= {}^{2}/_{3\pi}\sin({}^{3\pi}/_{4}) = \dfrac{\sqrt{2}}{3\pi} \\ C_4 &= {}^{2}/_{4\pi}\sin({}^{4\pi}/_{4}) = 0 \\ C_5 &= {}^{2}/_{5\pi}\sin({}^{5\pi}/_{4}) = -\dfrac{\sqrt{2}}{5\pi} \\ &\cdots \end{cases}$$

That is to say, function $x(t)$ can be synthesized as the sum of infinite sine series. Being a complex number $C_n = a_n + jb_n$, recall that $|C_n| = \sqrt{a_n^2 + b_n^2}$, thus

$$x(t) = \sum_{k=-\infty}^{\infty} C_n\ \exp(j\,n\omega_0 t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0\,t) + b_n\sin(n\omega_0\,t)$$

$$\{\text{for even functions } b_n = 0 \quad \therefore \quad |a_n| = |C_n|\}$$

Fig. 6.13 Example P.6.10

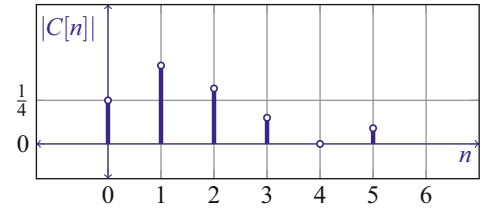
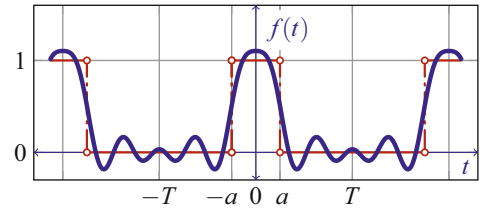


Fig. 6.14 Example P.6.10



$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\,\omega_0\, t) = \left\{\omega_0 = \frac{2\pi}{T},\quad a = \frac{T}{4}\right\}$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos\left(\frac{2\pi}{T}\,t\right) - \frac{1}{3}\cos\left(3 \times \frac{2\pi}{T}\,t\right) + \frac{1}{5}\cos\left(5 \times \frac{2\pi}{T}\,t\right) - \cdots\right]$$

As a comparison, this series is overlaid on $x(t)$, see Fig. 6.14 where only the first four terms are used, and spectrum is illustrated in Fig. 6.13.

6.11. **Method 1:** Given delayed (odd) pulse waveform, in $(0, T)$ period, coefficients of Fourier series may be calculated in sine/cosine form as

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{T}\int_0^{2a} (1)\,dt = \frac{4a}{T}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nt)\,dt = \frac{2}{T}\int_0^{2a} (1)\cos(nt)\,dt = \frac{2}{nT}\ \sin(nt)\Big|_0^{2a}$$

$$= \frac{2}{nT}\big(\sin(2a\,n) - \sin(0)\big) = \frac{2}{nT}\ \sin(2a\,n)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nt)\,dt = \frac{2}{T}\int_0^{2a} (1)\sin(nt)\,dt = -\frac{2}{nT}\ \cos(nt)\Big|_0^{2a}$$

$$= -\frac{2}{nT}\big(\cos(2a\,n) - \cos(0)\big) = \frac{2}{nT}\big(1 - \cos(2a\,n)\big)$$

Then, given specific high/low ratio and period $T = 2\pi$,
$a = T/4$: therefore,

$$a_0 = \frac{4a}{T} = \frac{\cancel{4}T}{T\cancel{4}} = 1$$

$$a_n = \frac{2}{nT}\sin(2a\,n) = \frac{2}{nT}\sin\left(\cancel{2}\frac{T}{\cancel{4}\,\cancel{2}}n\right) = \frac{2}{nT}\sin\left(\frac{T}{2}n\right) = \frac{\cancel{2}\,1}{\cancel{2}n\pi}\sin\left(\frac{\cancel{2}\pi}{\cancel{2}}n\right) = \frac{1}{n\pi}\sin(n\pi) \to 0$$

$$= 0$$

$$b_n = \frac{2}{nT}\left(1-\cos(2a\,n)\right) = \frac{2}{nT}\left(1-\cos\left(\cancel{2}\frac{T}{\cancel{4}\,\cancel{2}}n\right)\right) = \frac{1}{n\pi}\left(1-\cos(n\pi)\right)$$

A few calculated samples in the series are,

$$b_n = \frac{1}{n\pi}\left(1-\cos(n\pi)\right) = \begin{cases} b_1 & = {}^1\!/\!_\pi\left(1-\cos(\pi)\right) = \dfrac{2}{\pi} \\ b_2 & = {}^1\!/\!_{2\pi}\left(1-\cos(2\pi)\right) = 0 \\ b_3 & = {}^1\!/\!_{3\pi}\left(1-\cos(3\pi)\right) = \dfrac{2}{3\pi} \\ b_4 & = {}^1\!/\!_{4\pi}\left(1-\cos(4\pi)\right) = 0 \\ b_5 & = {}^1\!/\!_{5\pi}\left(1-\cos(5\pi)\right) = \dfrac{2}{5\pi} \\ & \cdots \end{cases}$$

Therefore, Fourier series of port function $a\,\Pi\left({}^{(t-\pi/2)}\!/\!_\pi\right)$ is

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} 0\,\cos(nt) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\pi}\left(1-\cos(n\pi)\right)\sin(nt)$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin(t) + 0 + \frac{2}{3\pi}\sin(3t) + 0 + \frac{2}{5\pi}\sin(5t) + \cdots$$

Note that delayed port function is *not* even and that even coefficients equal zero. The sum of first few terms of the Fourier series $f(t)$ is close approximation of an ideal square wave function, see Fig. 6.15.

Method 2: Relative to square waveform in P.6.9 function $f(t)$ in this example is time delayed, that is to say $y(t) = f(t-a)$. Knowing the time delay property of Fourier transform it can be written that

$$C_n = \exp(-jn\omega_0 a)\,\frac{1}{n\pi}\sin\left(\frac{2n\pi}{T}a\right)$$

and derive the same Fourier series as per **Method 1:** but faster.

6.12. Fourier series of delayed triangular function with period $T = 2\pi$ may be derived, for example, in sine/cosine form by definition.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\,t) + \sum_{n=1}^{\infty} b_n\sin(n\,t)$$

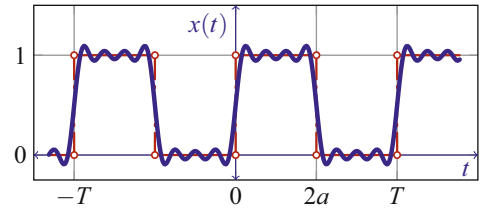


Fig. 6.15 Example P.6.11

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(n\,t)\ dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(n\,t)\ dt$$

Method 1: Given triangular function is periodic with period $T = 2\pi$. Note equations of linear sections within the chosen intervals. Without loosing generality of T, it may be written that

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(t)\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2a}{\pi}\,t\ dt + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\left(-\frac{2a}{\pi}\right)(t-\pi)\ dt$$

$$= \frac{2a}{\pi^2}\int_{-\pi/2}^{\pi/2} t\ dt - \frac{2a}{\pi^2}\int_{\pi/2}^{3\pi/2} t\ dt + \frac{2a}{\pi}\int_{\pi/2}^{3\pi/2} dt = \frac{2a}{\pi^2}\ t^2\Big|_{-\pi/2}^{\pi/2} - \frac{2a}{\pi^2}\ t^2\Big|_{\pi/2}^{3\pi/2} + \frac{2a}{\pi}\ t\Big|_{\pi/2}^{3\pi/2}$$

$$= \frac{2a}{\pi^2}\,\frac{1}{2}\left(\frac{\pi^2}{4} - \frac{(-\pi)^2}{4}\right) - \frac{2a}{\pi^2}\,\frac{1}{2}\left(\frac{9\pi^2}{4} - \frac{\pi^2}{4}\right) + \frac{2a}{\pi}\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$$

$$= -\frac{a}{\pi^2}\,\frac{8\ 2\ \pi^2}{4} + \frac{2a}{\pi}\,\pi = -a + a$$

$$= 0$$

As well, given period $T = 2\pi$,

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(t)\ \cos(n\,t)\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2a}{\pi}\,t\ \cos(n\,t)\ dt + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\left(-\frac{2a}{\pi}\right)(t-\pi)\ \cos(n\,t)\ dt$$

$$= \frac{2a}{\pi^2}\int_{-\pi/2}^{\pi/2} t\ \cos(n\,t)\ dt - \frac{2a}{\pi^2}\int_{\pi/2}^{3\pi/2} t\ \cos(n\,t)\ dt + \frac{2a}{\pi}\int_{\pi/2}^{3\pi/2}\cos(n\,t)\ dt$$

$$= \{\text{see Vol. II}\}$$

$$= \frac{2a}{n\pi^2}\ t\sin(n\,t)\Big|_{-\pi/2}^{\pi/2} + \frac{1}{n^2\pi^2}\ \cos(n\,t)\Big|_{-\pi/2}^{\pi/2} - \frac{2a}{n\pi^2}\ t\sin(n\,t)\Big|_{\pi/2}^{3\pi/2} - \frac{1}{n^2\pi^2}\ \cos(n\,t)\Big|_{\pi/2}^{3\pi/2}$$

$$
\begin{aligned}
&+\frac{2a}{n\pi}\ \sin(n\,t)\ \Big|_{\pi/2}^{3\pi/2}\\
&=\frac{2a}{n\pi^2}\left(\frac{\pi}{2}\sin\left(n\,\frac{\pi}{2}\right)+\frac{\pi}{2}\sin\left(-n\,\frac{\pi}{2}\right)\right)+\frac{1}{n^2\pi^2}\left(\cos\left(n\,\frac{\pi}{2}\right)-\cos\left(-n\,\frac{\pi}{2}\right)\right)\\
&\quad-\frac{2a}{n\pi^2}\left(\frac{3\pi}{2}\sin\left(n\,\frac{3\pi}{2}\right)-\frac{\pi}{2}\sin\left(n\,\frac{\pi}{2}\right)\right)-\frac{1}{n^2\pi^2}\left(\cos\left(n\,\frac{3\pi}{2}\right)-\cos\left(n\,\frac{\pi}{2}\right)\right)\\
&\quad+\frac{2a}{n\pi}\left(\sin\left(n\,\frac{3\pi}{2}\right)-\sin\left(n\,\frac{\pi}{2}\right)\right)\\
&=\left\{\frac{3\pi}{2}=-\frac{\pi}{2},\ \ \cos(-x)=\cos(x),\ \ \sin(-x)=-\sin(x)\right\}\\
&=-\frac{3a}{n\pi}\sin\left(-n\,\frac{\pi}{2}\right)+\frac{a}{n\pi}\sin\left(n\,\frac{\pi}{2}\right)+\frac{2a}{n\pi}\sin\left(-n\,\frac{\pi}{2}\right)-\frac{2a}{n\pi}\sin\left(n\,\frac{\pi}{2}\right)\\
&=\left(\frac{3a}{n\pi}-\frac{2a}{n\pi}\right)\sin\left(n\,\frac{\pi}{2}\right)-\left(\frac{2a}{n\pi}-\frac{a}{n\pi}\right)\sin\left(n\,\frac{\pi}{2}\right)\\
&=0
\end{aligned}
$$

and,

$$
\begin{aligned}
b_n&=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(t)\ \sin(n\,t)\ dt=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2a}{\pi}t\ \sin(n\,t)\ dt+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\left(-\frac{2a}{\pi}\right)(t-\pi)\ \sin(n\,t)\ dt\\
&=\frac{2a}{\pi^2}\int_{-\pi/2}^{\pi/2}t\ \sin(n\,t)\ dt-\frac{2a}{\pi^2}\int_{\pi/2}^{3\pi/2}t\ \sin(n\,t)\ dt+\frac{2a}{\pi}\int_{\pi/2}^{3\pi/2}\sin(n\,t)\ dt\\
&=\{\text{see Vol. II}\}\\
&=\frac{2a}{\pi^2}\left(-\frac{t}{n}\cos(nt)+\frac{1}{n^2}\sin(nt)\right)\Big|_{-\pi/2}^{\pi/2}-\frac{2a}{\pi^2}\left(-\frac{t}{n}\cos(nt)+\frac{1}{n^2}\sin(nt)\right)\Big|_{\pi/2}^{3\pi/2}\\
&\quad-\frac{2a}{n\pi}\cos(nt)\Big|_{\pi/2}^{3\pi/2}\\
&=\left\{\frac{3\pi}{2}=-\frac{\pi}{2},\ \ \cos(-x)=\cos(x),\ \ \sin(-x)=-\sin(x)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=-\frac{\pi}{2}\frac{2a}{n\pi^2}\cos(n\pi/2)+\frac{2a}{n^2\pi^2}\sin(n\pi/2)-\frac{\pi}{2}\frac{2a}{n\pi^2}\cos(n\pi/2)+\frac{2a}{n^2\pi^2}\sin(n\pi/2)\\
&\quad+\frac{3\pi}{2}\frac{2a}{n\pi^2}\cos(3n\pi/2)-\frac{2a}{n^2\pi^2}\sin(3n\pi/2)-\frac{\pi}{2}\frac{2a}{n\pi^2}\cos(n\pi/2)+\frac{2a}{n^2\pi^2}\sin(n\pi/2)\\
&\quad-\frac{2a}{n\pi}\cos(3n\pi/2)+\frac{2a}{n\pi}\cos(n\pi/2)\\
&=\frac{3a}{n\pi}\cos(n\pi/2)-\frac{3a}{n\pi}\cos(n\pi/2)+\frac{8a}{n^2\pi^2}\sin(n\pi/2)
\end{aligned}
$$

$$= \frac{8a}{n^2\pi^2}\ \sin(n\pi/2)$$

A few calculated terms in the series are

$$\frac{8a}{n^2\pi^2}\ \sin(n\pi/2) = \begin{cases} b_1 & = {}^{8a}/_{1^2\pi^2}\ \sin(\pi/2) = \dfrac{8a}{\pi^2} \\ b_2 & = {}^{8a}/_{2^2\pi^2}\ \sin(2\pi/2) = 0 \\ b_3 & = {}^{8a}/_{3^2\pi^2}\ \sin(3\pi/2) = -\dfrac{8a}{3^2\pi^2} \\ b_4 & = {}^{8a}/_{4^2\pi^2}\ \sin(4\pi/2) = 0 \\ b_5 & = {}^{8a}/_{5^2\pi^2}\ \sin(5\pi/2) = \dfrac{8a}{5^2\pi^2} \\ b_6 & = {}^{8a}/_{6^2\pi^2}\ \sin(6\pi/2) = 0 \\ b_7 & = {}^{8a}/_{6^2\pi^2}\ \sin(7\pi/2) = -\dfrac{8a}{7^2\pi^2} \\ & \ldots \end{cases}$$

which can be generalized as,

$$\frac{8a}{n^2\pi^2}\ \sin(n\pi/2) = = \begin{cases} 0; & n \ \text{ even} \\ \dfrac{8a}{(n\pi)^2}; & n = 1, 5, 9, \ldots \\ -\dfrac{8a}{(n\pi)^2}; & n = 3, 7, 11, \ldots \end{cases}$$

In summary,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\,t) = \frac{8a}{\pi^2}\left(\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) - \cdots\right)$$

A first few terms are superimposed over the ideal delayed triangle function $x(t)$ in Fig. 6.16.

Method 2: With a bit of practice and experience, one can take advantage of function's symmetry and summation of signed areas by definite integrals (see Calculus).

(a) By itself, delayed triangular function $x(t)$ is odd, see Fig. 6.17. Therefore, within a symmetric interval definite interval of $x(t)$ must equal zero. For example, an easy choice of symmetric interval is $-\pi/2, \pi/2$, so that

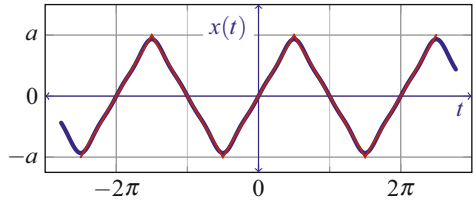
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{2a}{\pi}\ t\ dt = \frac{a}{\pi^2}\ t^2\ \Big|_{-\pi}^{\pi} = \frac{a}{\pi^2}\left(\cancel{(\pi/2)^2} - \cancel{(-\pi/2)^2}\right) = 0$$

This result is general for odd functions over symmetric interval

(b) The *product* of $x(t)\cos(nt)$ is also odd, see Fig. 6.17 (top). Definite integral of an odd function over symmetric interval must equal zero, which may be shown, for example as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(n\,t)\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{2a}{\pi}\ t\ \cos(n\,t)\ dt$$

Fig. 6.16 Example P.6.12



$$
\begin{aligned}
&= \{\text{see Vol. II}\} \\
&= \frac{2a}{n\pi^2}\ t\sin(n\,t)\,\Big|_{-\pi/2}^{\pi/2} + \frac{1}{n^2\pi^2}\ \cos(n\,t)\,\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{2a}{n\pi^2}\left(\frac{\pi}{2}\sin\left(n\frac{\pi}{2}\right) + \frac{\pi}{2}\sin\left(-n\frac{\pi}{2}\right)\right) + \frac{1}{n^2\pi^2}\left(\cos\left(n\frac{\pi}{2}\right) - \cos\left(-n\frac{\pi}{2}\right)\right) \\
&= 0
\end{aligned}
$$

(c) The *product* of $x(t)\sin(nt)$ is always positive, see Fig. 6.17 (bottom). By consequence, definite integral may be calculated as two times the integral on half the period. Thus,

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(n\,t)\ dt \\
&= \frac{2}{\pi}\int_0^{\pi/2}\frac{2a}{\pi}\,t\ \sin(n\,t)\ dt - \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{2a}{\pi}\,(t-\pi)\ \sin(n\,t)\ dt \\
&= \frac{4a}{\pi^2}\int_0^{\pi/2} t\ \sin(n\,t)\ dt - \frac{4a}{\pi^2}\int_{\pi/2}^{\pi} t\ \sin(n\,t)\ dt + \frac{4a}{\pi}\int_{\pi/2}^{\pi}\sin(n\,t)\ dt = \{\text{see Vol. II}\} \\
&= \frac{4a}{n^2\pi^2}\left(\sin(n\,t) - n\,t\cos(n\,t)\right)\Big|_0^{\frac{\pi}{2}} - \frac{4a}{n^2\pi^2}\left(\sin(n\,t) - n\,t\cos(n\,t)\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{4a}{n\pi}\ \cos(n\,t)\,\Big|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{4a}{n^2\pi^2}\Big(\big(\sin(n\,\pi/2) - n\,\pi/2\cos(n\,\pi/2)\big) - \big(\cancel{\sin(0)} - \cancel{0\cos(0)}\big)\Big) \\
&- \frac{4a}{n^2\pi^2}\Big(\big(\cancel{\sin(n\,\pi)} - n\,\pi\ \cancel{\cos(n\,\pi)}\big) - \big(\sin(n\,\pi/2) - n\,\pi/2\cos(n\,\pi/2)\big)\Big) \\
&- \frac{4a}{n\pi}\big(\cancel{\cos(n\,\pi)} - \cos(n\,\pi/2)\big) \\
&= \frac{4a}{n^2\pi^2}\sin(n\,\pi/2) + \frac{4a}{n^2\pi^2}\sin(n\,\pi/2) - \cancel{\frac{2a}{n\pi}\cos(n\,\pi/2)} + \cancel{\frac{4a}{n\pi}\cos(n\,\pi/2)} - \cancel{\frac{2a}{n\pi}\cos(n\,\pi/2)} \\
&= \frac{8a}{n^2\pi^2}\sin(n\,\pi/2)
\end{aligned}
$$

Which is the same result as by definition in **Method 1:** but with a bit less calculations.

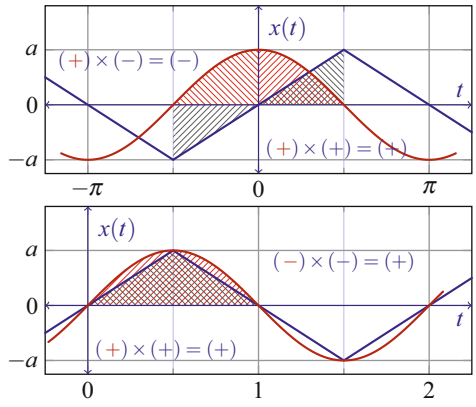


Fig. 6.17 Example P.6.12

6.3 Inverse Discrete Time FT

6.13. Given a simple sequence $X[k] = \{10, -2+2j, -2, -2-2j\}$ it follows that $N = 4$ and periodicity $T = 4$, therefore by definition

$$x[n] = \mathscr{F}^{-1}(X[k]) = \frac{1}{T}\sum_{-\infty}^{\infty} X[k]\,\exp\left(j\frac{2\pi k}{N}n\right) = \frac{1}{4}\sum_{n=0}^{3} X[k]\,\exp\left(j\frac{2\pi k}{4}n\right)$$

$$= \frac{1}{4}\Big[\,10\exp\left(j\frac{2\pi\ 0}{4}n\right) + (-2+j2)\,\exp\left(j\frac{2\pi\ 1}{4\ 2}n\right)$$

$$-2\,\exp\left(j\frac{2\pi\ 2}{4}n\right) + (-2-j2)\,\exp\left(j\frac{2\pi\ 3}{4\ 2}n\right)\Big]$$

$$\underline{\text{Case } n=0}:\ x[0] = \frac{1}{4}\Big[\,10+(-2+j2)\exp\left(j\frac{\pi}{2}\,0\right) - 2\exp\left(j\pi\ 0\right) + (-2-j2)\exp\left(j\frac{3\pi}{2}\,0\right)\Big]$$

$$= \frac{1}{4}\left[10-2+j2-2-2-j2\right] = 1$$

$$\underline{\text{Case } n=1}:\ x[1] = \frac{1}{4}\Big[\,10+(-2+j2)\exp\left(j\frac{\pi}{2}\,1\right) - 2\exp\left(j\pi\ 1\right) + (-2-j2)\exp\left(j\frac{3\pi}{2}\,1\right)\Big]$$

$$= \frac{1}{4}\Big[10+(-2+j2)(j) - 2(-1) + (-2-j2)(-j)\Big]$$

$$= \frac{1}{4}\Big[10-2j-2+2+2j-2\Big] = 2$$

$$\underline{\text{Case } n=2}: \; x[2] = \frac{1}{4}\left[\, 10+(-2+j2)\exp\left(j\frac{\pi}{2}2\right)^{-1} - 2\exp\left(j\pi\ 2\right)^{1} + (-2-j2)\exp\left(j\frac{3\pi}{2}\ 2\right)^{-1} \right]$$
$$= \frac{1}{4}\left[10 + (-2+j2)(-1) - 2(1) + (-2-j2)(-1)\right]$$
$$= \frac{1}{4}\left[10 + 2 - 2j - 2 + 2 + 2j\right] = 3$$

$$\underline{\text{Case } n=3}: \; x[3] = \frac{1}{4}\left[\, 10+(-2+j2)\exp\left(j\frac{\pi}{2}3\right)^{-j} - 2\exp\left(j\pi\ 3\right)^{-1} + (-2-j2)\exp\left(j\frac{3\pi}{2}\ 3\right)^{j} \right]$$
$$= \frac{1}{4}\left[10 + (-2+j2)(-j) - 2(-1) + (-2-j2)(j)\right]$$
$$= \frac{1}{4}\left[10 + 2j + 2 + 2 - 2j + 2\right] = 4$$

In conclusion, $x[n] = \{1, 2, 3, 4\}$.

6.14. Given a simple sequence $X[k] = \{5, 3-j2, -3, 3+j2\}$, it follows that $N = 4$ and periodicity $T = 4$, therefore by definition

$$x[n] = \mathscr{F}^{-1}(X[k]) = \frac{1}{T}\sum_{-\infty}^{\infty} X[k]\ \exp\left(j\frac{2\pi k}{N}n\right) = \frac{1}{4}\sum_{n=0}^{3} X[k]\ \exp\left(j\frac{2\pi k}{4\ 2}n\right)$$
$$= \frac{1}{4}\left[\, 5\exp\left(j\frac{2\pi\ 0}{4}n\right)^{1} + (3-j2)\ \exp\left(j\frac{2\pi\ 1}{4\ 2}n\right) \right.$$
$$\left. - 3\ \exp\left(j\frac{2\pi\ 2}{4}n\right) + (3+j2)\ \exp\left(j\frac{2\pi\ 3}{4\ 2}n\right) \right]$$

$$\underline{\text{Case } n=0}: \; x[0] = \frac{1}{4}\left[\, 5 + (3-j2)\ \exp\left(j\frac{\pi}{2}\,0\right) - 3\ \exp\left(j\pi\ 0\right) + (3+j2)\ \exp\left(j\frac{3\pi}{2}\,0\right) \right]$$
$$= \frac{1}{4}\left[5 + 3 - j2 - 3 + 3 + j2\right]$$
$$= 2$$

$$\underline{\text{Case } n=1}: \; x[1] = \frac{1}{4}\left[\, 5 + (3-j2)\exp\left(j\frac{\pi}{2}1\right)^{j} - 3\exp\left(j\pi\ 1\right)^{-1} + (3+j2)\exp\left(j\frac{3\pi}{2}\,1\right)^{-j} \right]$$
$$= \frac{1}{4}\left[5 + (3-j2)(j) - 3(-1) + (3+j2)(-j)\right]$$

$$= \frac{1}{4}\left[5 + 3j + 2 + 3 - 3j + 2\right]$$
$$= 3$$

$$\underline{\text{Case } n = 2} \;:\; x[2] = \frac{1}{4}\left[\, 5 + (3 - j2)\exp\left(j\frac{\pi}{2}2\right) - 3\exp(j\pi\ 2) + (3 + j2)\exp\left(j\frac{3\pi}{2}\ 2\right)\right]$$
$$= \frac{1}{4}\left[5 + (3 - j2)(-1) - 3(1) + (3 + j2)(-1)\right]$$
$$= \frac{1}{4}\left[5 - 3 + 2j - 3 - 3 - 2j\right] = -1$$

$$\underline{\text{Case } n = 3} \;:\; x[3] = \frac{1}{4}\left[\, 5 + (3 - j2)\exp\left(j\frac{\pi}{2}3\right) - 3\exp(j\pi\ 3) + (3 + j2)\exp\left(j\frac{3\pi}{2}\ 3\right)\right]$$
$$= \frac{1}{4}\left[5 + (3 - j2)(-j) - 3(-1) + (3 + j2)(j)\right]$$
$$= \frac{1}{4}\left[5 - 3j - 2 + 3 + 3j - 2\right] = 1$$

In conclusion, $x[n] = \{2, 3, -1, 1\}$.

6.4 Fast Fourier Transform

6.15. Discrete Fourier Transformation (DFT) of series $x[n]$ is by definition

$$\mathscr{F}_k(x[n]) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x[n]\exp\left(-j\frac{2\pi\ k}{N}\ n\right) \quad \text{where,} \quad k = 0, 1, 2, \ldots$$

In order to simplify the long equations writing, often exp() is used to denote the necessary exponential term in the above definition. As n, k, N changes, the number of necessary calculations increase as follows

$\underline{N = 1}$ $\mathscr{F}_0 = x[0]\ \exp()$ (one calculation)

$\underline{N = 2}$ $\mathscr{F}_0 = x[0]\ \exp() + x[1]\ \exp()$ (two calculations)

$\mathscr{F}_1 = x[0]\ \exp() + x[1]\ \exp()$ (two calculations)

(total: four calculations)

$\underline{N = 3}$ $\mathscr{F}_0 = x[0]\ \exp() + x[1]\ \exp() + x[2]\ \exp()$ (three calculations)

$\mathscr{F}_1 = x[0]\ \exp() + x[1]\ \exp() + x[2]\ \exp()$ (three calculations)

$$\mathscr{F}_2 = x[0]\ \exp() + x[1]\ \exp() + x[2]\ \exp() \quad \text{(three calculations)}$$

$$\overline{\text{(total: nine calculations)}}$$

$$\vdots$$

$$N \quad \Rightarrow \qquad \overline{\text{(total: } N^2 \text{ calculations)}}$$

That is to say, for example, the resolution of 1 Hz within 20 kHz bandwidth means that it is necessary to do $N = 20{,}000$ points in the transformation. Knowing that DFT calculation does N^2 operations, it follows that

$$t = N^2 \times 1\text{min} = \left(20 \times 10^3\right)^2 \text{ min} = 400 \times 10^6 \text{ min} \approx 278 \times 10^3 \quad \text{days}$$
$$\approx 760 \quad \text{years}$$

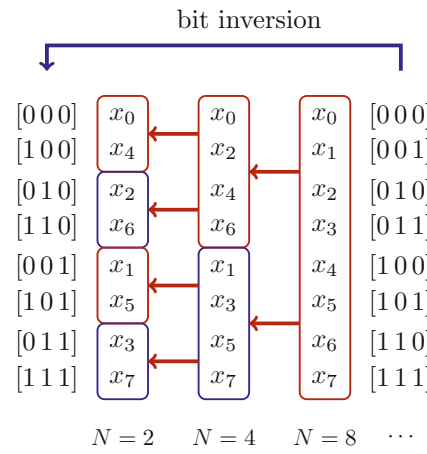
By comparison, FFT algorithm does $N \log_2 N$ calculations, that is to say

$$t = (N \log_2 N) \times 1\text{min} = \left(N \frac{\ln N}{\ln 2}\right) \times 1\text{min} = 286 \times 10^3 \text{ min} \approx 198 \quad \text{days}$$

The advantage of FFT over DFT increases as N becomes larger.

6.16. The main idea of FFT is to split the grand sum into two separate sums, in the first iteration to one with odd and one with even coefficients. In the subsequent iteration, again every second term is grouped as "odd' and the other half as "even" group, see Fig. 6.18. The process is repeated until each group consists of two terms only. This process is highly symmetric, as a consequence the binary form of aligned term indexes in the final iteration is binary inverted relative to the binary form of aligned term indexes at the beginning. Formally, FFT equations are rearranged as follows,

Fig. 6.18 Example P.6.16



$$\mathscr{F}_k(x[n]) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi\, k}{N}\, n\right)$$

$$= \underbrace{\sum_{m=0}^{N/2-1} x[2m] \exp\left(-j\frac{2\pi\, k}{N/2}\, \frac{2m}{2}\right)}_{\text{even index}} + \underbrace{\sum_{m=0}^{N/2-1} x[2m+1] \exp\left(-j\frac{2\pi\, k}{N/2}\, \frac{2m+1}{2}\right)}_{\text{odd index}}$$

$$\left\{C_k = \exp\left(-j\frac{2\pi\, k}{N}\right)\right\}$$

$$= \sum_{m=0}^{N/2-1} x[2m]\ \exp\left(-j\frac{2\pi\, k}{N/2}\, m\right) + C_k \sum_{m=0}^{N/2-1} x[2m+1]\ \exp\left(-j\frac{2\pi\, k}{N/2}\, m\right)$$

Where,

$$\exp\left(-j\frac{2\pi\, k}{N/2}\, m\right) = \cos\left(-\frac{2\pi\, k}{N/2}\, m\right) + j \sin\left(-\frac{2\pi\, k}{N/2}\, m\right); \quad (k = 0, 1, \ldots, N)$$

However, when $k > N/2$, that is to say $k = N/2 + r$ and $r = 1, 2, \ldots, N/2$, then

$$\cos\left(-\frac{2\pi\ (N/2+r)}{N/2}\, m\right) = \cos\left(\frac{-2\pi\ (N/2)\, m - 2\pi\, r\, m}{N/2}\right)$$

$$= \cos\left(-2\pi\, m - \frac{2\pi\, r\, m}{N/2}\right)$$

$$\{m = 0, 1, \ldots, N/2 \quad \therefore \quad -2\pi\, m = 0\}$$

$$= \cos\left(-\frac{2\pi\, r\, m}{N/2}\right) = \cos\left(-\frac{2\pi\ (N/2+r)\, m}{N/2}\right)$$

{known as "symmetry identity"}

This identity is valid for sine terms as well, which is to say that

$$\exp\left(-j\frac{2\pi\, k}{N/2}\, m\right) = \exp\left(-j\frac{2\pi\ (N/2+k)}{N/2}\, m\right); \quad (k = 0, 1, 2, \ldots, N)$$

As a consequence, it is sufficient to cary out only $N/2$ calculations. However, each of the summing groups can be split again and again, thus the total number of calculations is $N \log_2 N$.

6.17. Given $N = 2$ and a general discrete series $x_n = x[n]$, Fourier transform $\mathscr{F}_k(x[n]) = X_k$ equations are written by using following syntax,

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-j\frac{2\pi}{N}\, n\, k\right) = \sum_{n=0}^{N-1} x_n\ w_N^{n\cdot k} = \{N = 2\}$$

$$= \sum_{n=0}^{1} x_n \; w_2^{n \cdot k}$$

This syntax simplifies the repetitive writing of complex exponential terms, keep in mind that $w_N^{n \cdot k}$ represents the following calculations

$$w_N \equiv \exp\left(-j\frac{2\pi}{N}\right) \;\; \therefore \;\; \exp\left(-j\frac{2\pi}{N}\, n\, k\right) = \left[\exp\left(-j\frac{2\pi}{N}\right)\right]^{n \cdot k} \stackrel{\text{def}}{=} w_N^{n \cdot k}$$

where the power term is calculated as the product nk, for example, if $N = 2$ it follows that

$$w_2^{0 \cdot 0} = \left[\exp\left(-j\frac{\cancel{2}\pi}{\cancel{2}}\right)\right]^{0 \cdot 0} = \left[\exp\left(-j\pi\right)\right]^{0 \cdot 0} = (-1)^0 = 1$$

or,

$$w_2^{1 \cdot 1} = \left[\exp\left(-j\frac{\cancel{2}\pi}{\cancel{2}}\right)\right]^{1 \cdot 1} = \left[\exp\left(-j\pi\right)\right]^{1 \cdot 1} = (-1)^1 = -1$$

For the moment, let us keep $n > 0$ exponential terms in place, and also use the identity $w_2^{1 \cdot 1} \equiv -w_2^{0 \cdot 0}$, even if their values are as simple as $+1$ or -1. Reason for this choice becomes evident as $N = 4$, $N = 8$, etc.. In this case,

k=0: $\quad X[0] = x_0\, w_2^{0 \cdot 0} + x_1\, w_2^{1 \cdot 0} = x_0 + x_1 \equiv x_0 + x_1 \; w_2^0$

k=1: $\quad X[1] = x_0\, w_2^{0 \cdot 1} + x_1\, w_2^{1 \cdot 1} = x_0 - x_1 \equiv x_0 - x_1 \; w_2^0$

This calculation process is illustrated by "butterfly diagram", Fig. 6.19. The solutions X_k on the right side are formed by the graph inspection. For example, x_0 is first multiplied by "1" then added to $x_1\, w_2^0$ at the summing node to produce $X[0] = x_0 + x_1\, w_2^0$. Similarly, graph illustrates the formation of $X[1]$ equation. This basic butterfly diagram is then reused for any $N = 2$ FFT. As illustrated by the colour scheme, note paths that carry '$+1$' multiplication factor, paths that carry positive exponential factor, and paths that carry negative exponential factor.

6.18. In the case of $N = 4$, the four X_k DFT transformation are reorganized into symmetric form that can be directly translated into butterfly diagram as follows. As a reminder, shorthand syntax for $N = 4$ is

$$w_4 \equiv \exp\left(-j\frac{2\pi}{4}\right) = \exp\left(-j\frac{\pi}{2}\right) \;\; \therefore \;\; \exp\left(-j\frac{\pi}{2}\right)^{n\,k} \equiv w_4^{n \cdot k}$$

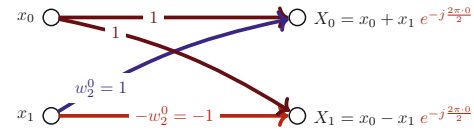


Fig. 6.19 Example P.6.17

$$\therefore$$

$$w_4^0 = \exp\left(-j\frac{\pi}{2}\right)^0 = 1$$

$$w_4^1 = \exp\left(-j\frac{\pi}{2}\right)^1 = -j$$

$$w_4^2 = \exp\left(-j\frac{\pi}{2}\right)^2 = \exp\left(-j\pi\right) = -1$$

$$w_4^3 = \exp\left(-j\frac{\pi}{2}\right)^3 = \exp\left(-j\frac{3\pi}{2}\right) = \exp\left(j\frac{\pi}{2}\right) = j$$

so that,

$$\begin{aligned}
\underline{k=0:}\quad X_0 &= x_0\, w_4^{0\cdot 0} + x_1\, w_4^{1\cdot 0} + x_2\, w_4^{2\cdot 0} + x_3\, w_4^{3\cdot 0} \\
&= x_0 + x_1 + x_2 + x_3 \quad (\text{“DC component”}) \\
\underline{k=1:}\quad X_1 &= x_0\, w_4^{0\cdot 1} + x_1\, w_4^{1\cdot 1} + x_2\, w_4^{2\cdot 1} + x_3\, w_4^{3\cdot 1} \\
&= x_0 + x_1\, w_4^1 + x_2\, w_4^2 + x_3\, w_4^3 = \left\{\frac{3\pi}{2} \equiv -\frac{\pi}{2}\right\} \\
&= x_0 + x_1\, w_4^1 - x_2 - x_3\, w_4^1
\end{aligned}$$

$$\begin{aligned}
\underline{k=2:}\quad X_2 &= x_0\, w_4^{0\cdot 2} + x_1\, w_4^{1\cdot 2} + x_2\, w_4^{2\cdot 2} + x_3\, w_4^{3\cdot 2} \\
&= x_0 + x_1\, w_4^2 + x_2\, w_4^4 + x_3\, w_4^6 \\
&= x_0 - x_1 + x_2 - x_3 \\
\underline{k=3:}\quad X_3 &= x_0\, w_4^{0\cdot 3} + x_1\, w_4^{1\cdot 3} + x_2\, w_4^{2\cdot 3} + x_3\, w_4^{3\cdot 3} \\
&= x_0 + x_1\, w_4^3 + x_2\, w_4^6 + x_3\, w_4^9 = \left\{\frac{9\pi}{2} \equiv -\frac{3\pi}{2}\right\} \\
&= x_0 + x_1\, w_4^1 - x_2 - x_3\, w_4^1
\end{aligned}$$

This set of equation may be written in a more symmetric form as,

$$\begin{aligned}
X_0 &= (x_0 + x_2) + w_4^0(x_1 + x_3) \\
X_1 &= (x_0 - x_2) + w_4^1(x_1 - x_3) \\
X_2 &= (x_0 + x_2) + w_4^2(x_1 + x_3) \\
X_3 &= (x_0 - x_2) + w_4^3(x_1 - x_3)
\end{aligned}$$

which can be further structured to explicitly include the form of two-bit butterfly (i.e. $N = 2$) as

$$X_0 = (x_0 + w_2^0\, x_2) + w_4^0(x_1 + w_2^0\, x_3)$$

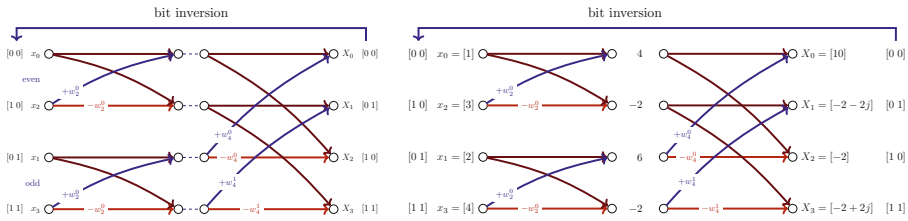


Fig. 6.20 Examples P.6.18, P.6.19

$$X_1 = (x_0 - w_2^0\, x_2) + w_4^1 (x_1 - w_2^0\, x_3)$$
$$X_2 = (x_0 + w_2^0\, x_2) - w_4^0 (x_1 + w_2^0\, x_3)$$
$$X_3 = (x_0 - w_2^0\, x_2) - w_4^1 (x_1 - w_2^0\, x_3)$$

Direct graph implementation of these equations is illustrated in Fig. 6.20 (left), where starting from the left side of the graph and following x_n terms along the associated paths (the addition operations are done at the merging nodes) the four X_k equations are composed simply by inspection. Note the index order of x_n and X_k indexes in this graph structure.

6.19. In order to use FFT graph diagram, in the first step x_n terms are reorganized on the left side by using the index bit inversion relative to X_k, see Fig. 6.20 (right). The intermediate sums are calculated using the two-bit grouping on the left side, then the second group of addition/multiplication operations is done by following paths in the right side of the graph.

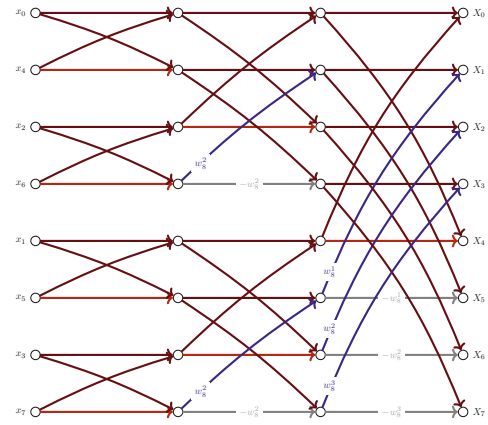
6.20. In the case of $N = 8$, shorthand syntax is

$$w_8 \equiv \exp\left(-j\frac{2\pi}{8}\right) = \exp\left(-j\frac{\pi}{4}\right) \quad \therefore \quad \exp\left(-j\frac{\pi}{4}\right)^{n\,k} \equiv w_8^{n\cdot k}$$

$\therefore$

$$w_8^0 = \exp\left(-j\frac{\pi}{4}\right)^0 = 1$$
$$w_8^1 = \exp\left(-j\frac{\pi}{4}\right)^1 = \frac{\sqrt{2}}{2}(1-j)$$
$$w_8^2 = \exp\left(-j\frac{\pi}{4}\right)^2 = \exp\left(-j\frac{\pi}{2}\right) = -j$$
$$w_8^3 = \exp\left(-j\frac{\pi}{4}\right)^3 = \exp\left(-j\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}(1+j)$$
$$w_8^4 = \exp\left(-j\frac{\pi}{4}\right)^4 = \exp\left(-j\pi\right) = -1 \equiv -w_8^0$$

Fig. 6.21 Example P.6.20



$$w_8^5 = \exp\left(-j\frac{\pi}{4}\right)^5 = \exp\left(-j\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}(1-j) \equiv -w_8^1$$

$$w_8^6 = \exp\left(-j\frac{\pi}{4}\right)^6 = \exp\left(-j\frac{6\pi}{4}\right) = j \equiv -w_8^2$$

$$w_8^7 = \exp\left(-j\frac{\pi}{4}\right)^7 = \exp\left(-j\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}(1+j) \equiv -w_8^3$$

In reference to Fig. 6.18, FFT graph flow is derived for $N = 8$ as in Fig. 6.21, where horizontal red lines imply multiplication by '-1', dark lines imply multiplication by '1', and other multiplication factors are written explicitly.

6.21. Given sequence $x_n = [1, 1, -1 - 1, 1, 1, -1 - 1]$, by inspection of graph in Fig. 6.21, where horizontal red lines imply multiplication by '-1' it follows that,

$$\begin{aligned} X_0 &= x_0 + x_4 + x_2 + x_6 + x_1 + x_5 + x_3 + x_7 = 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} X_1 &= x_0 - x_4 + (x_2 - x_6)(-j) + \Big(x_1 - x_5 + (x_3 - x_7)(-j)\Big)\, w_8^2 \\ &= 1 - 1 - 1 + 1 + (1 - 1 + (-1 + 1)(-j))w_8^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} X_2 &= x_0 + x_4 + (x_2 + x_6)(-1) + \Big(x_1 + x_5 + (x_3 + x_7)(-1)\Big)\,(-j) \\ &= 1 + 1 + 1 + 1 + (1 + 1 + 1 + 1)(-j) \\ &= 4 - 4j \end{aligned}$$

$$X_3 = x_0 - x_4 + (x_2 - x_6)(-w_8^2) + x_1 - x_5 + (x_3 - x_7)(-w_8^2)$$

$$
\begin{aligned}
&= 1 - 1 + (-1 + 1)(-w_8^2) + 1 - 1 + (-1 + 1)(-w_8^2) \\
&= 0 \\
X_4 &= x_0 + x_4 + x_2 + x_6 + (x_1 + x_5 + x_3 + x_7)(-1) \\
&= 1 + 1 - 1 - 1 - 1 - 1 + 1 + 1 \\
&= 0 \\
X_5 &= x_0 - x_4 + (x_2 - x_6)(-j) + \Big(x_1 - x_5 + (x_3 - x_7)(-j)\Big)(-w_8^1) \\
&= 1 - 1 + (-1 + 1)(-j) + \Big(1 - 1 + (-1 + 1)(-j)\Big)(-w_8^1) \\
&= 0 \\
X_6 &= x_0 + x_4 + (x_2 + x_6)(-1) + \Big(x_1 + x_5 + (x_3 + x_7)(-1)\Big)(j) \\
&= 1 + 1 + 1 + 1 + (1 + 1 + 1 + 1)(j) \\
&= 4 + 4j \\
X_7 &= x_0 + x_4 + (x_2 - x_6)(j) + \Big(x_1 - x_5 + (x_3 - x_7)(j)\Big)(-w_8^3) \\
&= 1 + 1 + (-1 + 1)(j) + \Big(1 - 1 + (-1 + 1)(j)\Big)(-w_8^3) \\
&= 0
\end{aligned}
$$

The total number of sum/product calculations is $8 \log_2 8 = 24$ in comparison with DFT where the total number of calculations equals $8^2 = 64$.

Laplace Transformation

7

Laplace transformation (LT) is a tool to convert time domain function $f(t)$ into its s–domain equivalent $F(s)$. Virtually, all books on Laplace transformation first present table with $f(t) \to F(s)$ LT pairs of basic functions. However, it is very important that, instead of memorizing these tabulated pairs, these $f(t) \to F(s)$ pairs are derived from definition. After some practice and experience, the lookup tables are used to speed up the calculations.

Laplace and inverse Laplace integrals: given complex variable $s = \sigma + j\omega$, $(\sigma, \omega \in \mathbb{R})$, then unilateral LT and inverse LT integrals are

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{+\infty} f(t)\, e^{-st}\, dt \tag{7.1}$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{j\,2\pi} \lim_{\tau \to \infty} \int_{c-j\tau}^{c+j\tau} F(s)\, e^{st}\, ds \tag{7.2}$$

where $c \in \mathbb{R}$ and "contour integral" (7.2) is solved with the help of the residue theorem. Note that solution to inverse LT my not be unique. Traditionally, the inverse LT integral is not solved at the introductory level courses; instead, "the table of simple inverse transforms" and LT properties are used.

Properties of Laplace Transformation

1. *Linearity:* given three functions $f(t)$, $x(t)$, and $y(t)$ and constants a, b, then

$$f(t) = ax(t) + by(t) \quad \Rightarrow \quad \mathscr{L}\{f(t)\} = a\mathscr{L}\{x(t)\} + b\mathscr{L}\{y(t)\}$$

2. *Time shifting:* given $x(t) \overset{\text{LT}}{\to} X(s)$, a constant $a > 0$, and step function $u(t)$, then

$$\mathscr{L}\{x(t-a)u(t-a)\} = e^{-as} X(s)$$

3. *Frequency shifting:* given $x(t) \overset{\text{LT}}{\to} X(s)$ and a constant $a > 0$, then

$$\mathscr{L}\left\{e^{at} x(t)\right\} = X(s-a)$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5_7

4. *Time scaling:* given $x(t) \overset{\text{LT}}{\to} X(s)$ and a constant $a > 0$, then

$$\mathscr{L}\{x(at)\} = \frac{1}{a} X\left(\frac{s}{a}\right)$$

5. *Derivative property:* given n times differentiable $x(t) \overset{\text{LT}}{\to} X(s)$, then

$$\mathscr{L}\left\{x^{(n)}\right\} = s^n X(s) - \sum_{k=1}^{n} x^{k-1}(0)$$

6. *Multiplication by t^n:* given $x(t) \overset{\text{LT}}{\to} X(s)$ and a positive integer $n > 0$, then

$$\mathscr{L}\left\{t^n\, x(t)\right\} = (-1)^n X^{(n)}(s)$$

7. *Convolution property:* given functions $x(t) \overset{\text{LT}}{\to} X(s)$ and $y(t) \overset{\text{LT}}{\to} Y(s)$, then

$$\mathscr{L}\{x(t) * y(t)\} = X(s)\, Y(s)$$

Problems

7.1 Basic Laplace Transformations

Given functions in P.7.1 to P.7.15, derive their respective LT by using definition integral (7.1). Assume $a > 0, \quad a \in \mathbb{R}$.

7.1. $f(t) = 1$ **7.2.** $f(t) = u(t)$ **7.3.** $f(t) = u(t-a)$

7.4. $f(t) = \delta(t)$ **7.5.** $f(t) = \delta(t-a)$ **7.6.** $f(t) = t\,u(t)$

7.7. $f(t) = t^2\,u(t)$ **7.8.** $f(t) = t^a\,u(t)$ **7.9.** $f(t) = e^{at}\,u(t)$

7.10. $f(t) = e^{-at}\,u(t)$ **7.11.** $f(t) = \left(e^{at} + e^{-at}\right)\,u(t)$ **7.12.** $f(t) = u(t) - u(t-a)$

7.13. $f(t) = \sin(at)\,u(t)$ **7.14.** $f(t) = \cos(at)\,u(t)$ **7.15.** $\forall t \in \{\mathbb{R}\} : f(t) = f(t+T)$

7.2 Basic Properties

It is educative, instead of memorizing the properties table, to prove LT properties starting from the definition integral. After some practice, these properties may be applied directly and speed up the calculations.

Given functions in P.7.16 to P.7.30, where applicable, derive properties of Laplace transformation assuming $t \geq 0$ and $a > 0$.

7.16. $g(t) = f'(t)$ **7.17.** $g(t) = f''(t)$ **7.18.** $g(t) = f'''(t)$

7.19. $f(t) = \sin(2t)$ **7.20.** $f(t) = \cos(2t)$ **7.21.** $f(t) = t\sin(2t)$

7.22. $f(t) = t^2 \cos t$ **7.23.** $g(t) = e^{at} f(t)$ **7.24.** $g(t) = f(at)$

7.25. $g(t) = \dfrac{f(t)}{t}$ **7.26.** $g(t) = \displaystyle\int_0^t f(z)\,dz$ **7.27.** $f(t) = \displaystyle\int_0^t (p^2 + p - e^p)\,dp$

7.28. $f(t) = \dfrac{e^{4t} - e^{-3t}}{t}$ **7.29.** $f(t) = (t-2)^3\,u(t-2)$ **7.30.** $f(t) = \dfrac{\sin t}{t}\,u(t)$

7.3 Inverse Laplace Transformation

Given in P.7.31 to P.7.42, derive their respective inverse LT by taking advantage of already derived transformation pairs in the previous examples of this chapter.

7.31. $\dfrac{1}{s}$ **7.32.** $\dfrac{1}{s}e^{-2s}$ **7.33.** 1

7.34. e^{-s} **7.35.** $\dfrac{1}{s^2}$ **7.36.** $\dfrac{2}{s^3}$

7.37. $\dfrac{s}{s^2+9}$ **7.38.** $\dfrac{1}{s}\left(1 - e^{-2s}\right)$ **7.39.** $\dfrac{4s}{(s^2+4)^2}$

7.40. $\arctan\left(\dfrac{1}{s}\right)$ **7.41.** $\dfrac{2s}{s^2 - a^2}$ **7.42.** $\dfrac{s}{s^2+4s+5}$

7.4 Differential Equations

Differential equations include the solution function as well as its derivatives. Very efficient method for solving some classes of differential equations is to apply "derivative property" of LT and convert these equations into ordinary algebraic polynomials. Even without formal (inductive) proof, it should not be difficult to adopt this property and use it within specific problems.

Reminder:: "derivative property" of LT is

$$F_n(s) = \mathscr{L}\left\{f^{(n)}(t)\right\} = s^n \cdot \mathscr{L}\left\{f(t)\right\} - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

where s^n is nth power of the polynomial variable s, $f^{(n)}(t)$ stands for nth derivative of $f(t)$, and $f(0)$ stands for the initial condition of $f(t)$ at $t = 0$, i.e., *before* the input signal is injected.

By applying LT derivative property, solve differential equations given in P.7.43 to P.7.51. As already known, ordinary integrals deliver general solution, i.e., there is an infinity of solution functions separated by a constant. Given sufficient number (equal to the order of differential equation) of the initial conditions, general solution is resolved into specific solution.

7.43. $y' + 1 = 0, \quad y(0) = 0$

7.44. $-2y' + y = 0, \quad y(0) = 1$

7.45. $y'' - 1 = 0, \quad y'(0) = 0, y(0) = 0$

7.46. $y'' - y' + 1 = 0, \quad y'(0) = 1, y(0) = 1$

7.47. $y'' + 2y' + 2y = 0, \quad y'(0) = 1, y(0) = -1$

7.48. $y'' + y = \sin(2t), \quad y'(0) = 0, y(0) = 0$

7.49. $y'' + y' = 2e^{-t}, \quad y'(0) = 1, y(0) = 1$

7.50. $y'' - y' = \cos(t), \quad y(0) = 0, y'(0) = 0$

7.51. $y''' - y'' = 0, \quad y(0) = -3, y'(0) = 1, y''(0) = 4$

Answers

Tutorial-type solutions to given examples are aligned so that subsequent forms are, as much as possible, logical development from the preceding ones. Thus, it is highly recommended, before looking into the proposed solutions, to make every effort and to explore own ideas. What is more, as they are backward referenced in the subsequent examples, it is assumed that the previously introduced ideas are already adopted. After reading the solutions, it is important to independently try again and even to explore other possible methods. Finally, as each problem could be solved with several different techniques, the methods presented in each of the problems are by no means the only ones possible.

7.1 Basic Laplace Transformations

7.1. Given time domain function

$$f(t) = 1$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} 1\, e^{-st}\, dt = \{\text{change variable}\}$$

$$\left\{\underline{z = -st} \quad \therefore \quad \frac{dz}{dt} = -s \quad \therefore \quad dt = -\frac{1}{s} dz\right\}$$

$$= -\frac{1}{s} \int e^z\, dz = -\frac{1}{s}\, e^z \ \{\text{back to the original variable } 's' \text{and its limits}\}$$

$$= -\frac{1}{s}\, e^{-st}\Big|_0^{+\infty} = -\frac{1}{s}\left[\cancel{e^{-\infty}}^{0} - \cancel{e^{0}}^{1}\right] = \frac{1}{s}$$

Note that, given $t \geq 0$, a constant "1" is indistinguishable from step function $u(t)$.

7.2. Given time domain function of unit step

$$f(t) = u(t)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} u(t)\, e^{-st}\, dt = \{u(t) = 1 \quad \text{for} \quad t \geq 0\} = \int_0^{+\infty} 1\, e^{-st}\, dt$$

$$= \{\text{see P.7.1}\} = \frac{1}{s}$$

which is expected, considering that for $x \geq 0$ function $u(t) = 1$.

7.3. Given time domain function of delayed unit step

$$f(t) = u(t-a)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} u(t-a)\, e^{-st}\, dt = \{u(t-a) = 1 \quad \text{for} \quad t \geq a\}$$

$$= \int_0^a 0\, e^{-st}\, dt + \int_a^{+\infty} 1\, e^{-st}\, dt = -\frac{1}{s}\int e^z\, dz$$

$$\left\{\text{change variable: } \underline{z = -st} \quad \therefore \quad \frac{dz}{dt} = -s \quad \therefore \quad dt = -\frac{1}{s}dz\right\}$$

$$= -\frac{1}{s}\, e^{-st}\Big|_a^{+\infty} = -\frac{1}{s}\left(e^{-\infty} - e^{-as}\right) = \frac{1}{s}\, e^{-as}$$

Compare this result with P.7.2 and note that time–delaying a function by a translates into multiplying its Laplace transformation with the exponential term $\exp(-as)$. In general, this case is known as "time shifting" property of Laplace transformation.

Reminder: Given that $\mathscr{L}\{f(t)\} = F(s)$, time shifting property of Laplace transformation is generalized as

$$\mathscr{L}\{f(t-a)\} = e^{-as} F(s)$$

7.4. Given time domain Dirac delta function

$$f(t) = \delta(t)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} \delta(t)\, e^{-st}\, dt = \{t = 0\} = e^0 \int_0^{+\infty} \delta(t)\, dt$$

$$= 1$$

because, at $t = 0$ Dirac delta function is included within the $(0, +\infty)$ interval.

7.5. Given time–delayed Dirac delta function

$$f(t) = \delta(t-a)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} \delta(t-a)\, e^{-st}\, dt = \{t = a, a > 0\} = e^{-as} \int_0^{+\infty} \delta(t-a)\, dt \;\; (\to 1)$$
$$= e^{-as}$$

compare with the result in P.7.4 (as expected and already noted in P.7.3).

7.6. Given time domain ramp function

$$f(t) = t\, u(t)$$

Method 1: by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} t\, u(t)\, e^{-st}\, dt = \{u(t) = 1 \quad \text{for} \quad t \geq 0\} = \int_0^{+\infty} t\, e^{-st}\, dt$$
$$= \left\{ z = -st \;\; \therefore \;\; \frac{dz}{dt} = -s \;\; \therefore \;\; dt = -\frac{1}{s}\, dz, \quad \text{and} \quad t = -\frac{1}{s}\, z \right\}$$
$$= \frac{1}{s^2} \int z\, e^z\, dz = \{\text{integration by parts: } u = z, \;\; dv = e^z dz \;\; \therefore \;\; du = dz, \;\; v = e^z\}$$
$$= \frac{1}{s^2}\left(z\, e^z - \int e^z\, dz\right) = \frac{1}{s^2}\left(z\, e^z - e^z\right) = \frac{1}{s^2}\left(e^z(z-1)\right)$$
$$\{\text{return to the original variable}\}$$
$$= \frac{1}{s^2}\left(e^{-st}(-st-1)\right)\Big|_0^{+\infty} = \frac{1}{s^2}\left(0 - e^0(-(0)s - 1)\right) = \frac{1}{s^2}$$

Method 2: Compare with the result in P.7.2 and note the consequence of multiplying $u(t)$ by t. In general, this case is known as "multiplying with power of t" property of Laplace transformation.

Reminder: Given that $\mathscr{L}\{f(t)\} = F(s)$, multiplying with power of t property of Laplace transformation is generalized as

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

where, $F^{(n)}(s)$ stands for nth derivative of $F(s)$.

In this example, power of t^1 is $n = 1$, so multiplying with power of t property is applied directly as

$$f(t) = u(t) \;\; \therefore \;\; F(s) = \frac{1}{s} \;\; \therefore \;\; F^{(1)}(s) = \frac{dF(s)}{ds} = -\frac{1}{s^2} \quad \text{and} \quad (-1)^1 = -1$$

$\therefore$

$$\mathscr{L}\{t\,u(t)\} = (-1)^1 \frac{dF(s)}{ds} = -1\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$$

7.7. Given $u(t)$ multiplied by t^2, as

$$f(t) = t^2\,u(t)$$

Method 1: by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$\begin{aligned}
F(s) &= \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{+\infty} t^2\,u(t)\,e^{-st}\,dt = \{u(t) = 1 \quad \text{for} \quad t \geq 0\} = \int_0^{+\infty} t^2\,e^{-st}\,dt \\
&= \left\{z = -st \;\;\therefore\;\; \frac{dz}{dt} = -s \;\;\therefore\;\; dt = -\frac{1}{s}\,dz, \quad \text{and} \quad t^2 = \frac{1}{s^2}\,z^2\right\} \\
&= -\frac{1}{s^3}\int z^2\,e^z\,dz = \{\text{integration by parts: } u = z^2, \;\; dv = e^z dz \;\;\therefore\;\; du = 2z\,dz, \;\; v = e^z\} \\
&= -\frac{1}{s^3}\left(z^2\,e^z - 2\int z\;e^z\,dz\right) \quad \left\{\text{see P.7.6 : } \int z\;e^z\,dz = e^z(z-1)\right\} \\
&= -\frac{1}{s^3}\left(z^2\,e^z - 2e^z(z-1)\right) = -\frac{1}{s^3}e^z(z^2 - 2z + 2) \;\{\text{return to the original variable}\} \\
&= -\frac{1}{s^3}\left[e^{-st}\left((-st)^2 - 2(-st) + 2\right)\right]_0^{+\infty} = -\frac{1}{s^3}\left[0 - 1(0 + 0 + 2)\right] = \frac{2}{s^3}
\end{aligned}$$

Method 2: By using the "multiplying with power of t" property of Laplace transformation, in this example, power of t^2 is $n = 2$, so multiplying with power of t property is applied directly as

$$f(t) = u(t) \;\;\therefore\;\; F(s) = \frac{1}{s} \;\;\therefore\;\; F^{(2)}(s) = \frac{d^2F(s^2)}{ds} = \frac{2}{s^3} \quad \text{and} \quad (-1)^2 = 1$$

$$\therefore$$

$$\mathscr{L}\{t\,u(t)\} = (-1)^2 \frac{d^2F(s)}{ds^2} = 1\left(\frac{2}{s^3}\right) = \frac{2}{s^3}$$

Evidently, with the experience, the properties of Laplace transformation are used as a shortcut.

7.8. Given differential equation with constant coefficients

$$f(t) = t^a\,u(t)$$

without knowing value of a, evidently, solving by definition is not much practical because it requires a bit longer proof by the induction method.

Reminder: Positive integer powers of t^a are generalized by the induction method as

$$\mathscr{L}\left\{t^a u(t)\right\} = (-1)^a u^{(a)}(s) = \frac{a!}{s^{a+1}}$$

where, $u^{(a)}(s)$ stands for ath derivative of $u(s)$.

Therefore,

$$F(s) = \int_0^{+\infty} t^a\, e^{-st}\, dt = \frac{a!}{s^{a+1}}$$

7.9. Given time domain function

$$f(t) = e^{at}\, u(t)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\left\{f(t)\right\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} e^{at}\, e^{-st}\, dt = \int_0^{+\infty} e^{-(s-a)t}\, dt = \left\{\text{change variable}\right\}$$

$$\left\{\underline{z = -(s-a)t} \quad \therefore \quad \frac{dz}{dt} = -(s-a) \quad \therefore \quad dt = -\frac{1}{s-a}dz\right\}$$

$$= -\frac{1}{s-a}\int e^z\, dz = -\frac{1}{s-a}\, e^z \left\{\text{return to the original variable}\right\}$$

$$= -\frac{1}{s-a}\, e^{-(s-a)t}\Big|_0^{+\infty} = -\frac{1}{s-a}\left[\cancel{e^{-\infty}}^{\,0} - \cancel{e^{0}}^{\,1}\right] = \frac{1}{s-a}$$

Note, if

$$\underline{a = 1} \;:\; F(s) = \int_0^{+\infty} e^t\, e^{-st}\, dt = \frac{1}{s-1}$$

$$\underline{a = -1} \;:\; F(s) = \int_0^{+\infty} e^{-t}\, e^{-st}\, dt = \frac{1}{s+1}$$

7.10. Given time domain function

$$f(t) = e^{-at}\, u(t)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\left\{f(t)\right\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} e^{-at}\, e^{-st}\, dt = \int_0^{+\infty} e^{-(s+a)t}\, dt = \left\{\text{change variable}\right\}$$

$$\left\{\underline{z = -(s+a)t} \quad \therefore \quad \frac{dz}{dt} = -(s+a) \quad \therefore \quad dt = -\frac{1}{s+a}dz\right\}$$

$$= -\frac{1}{s+a}\int e^z\,dz = -\frac{1}{s+a}\,e^z \quad \{\text{return to the original variable}\}$$

$$= -\frac{1}{s+a}\,e^{-(s+a)t}\Big|_0^{+\infty} = -\frac{1}{s+a}\left[e^{-\infty} - e^{0}\right] = \frac{1}{s+a}$$

7.11. Given time domain function

$$f(t) = e^{at}\,u(t) + e^{-at}\,u(t)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{+\infty}\left(e^{at}+e^{-at}\right)e^{-st}\,dt = \int_0^{+\infty} e^{-(s-a)t}\,dt + \int_0^{+\infty} e^{-(s+a)t}\,dt$$

$\{$see P.7.9 and P.7.10$\}$

$$= \frac{1}{s-a} + \frac{1}{s+a} = \frac{(s+a)+(s-a)}{(s-a)(s+a)} = \frac{2s}{s^2-a^2}$$

7.12. Given time domain function of rectangular impulse

$$f(t) = u(t) - u(t-a)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{+\infty}\big(u(t)-u(t-a)\big)\,e^{-st}\,dt = \int_0^{+\infty} u(t)\,e^{-st}\,dt$$

$$-\int_0^{+\infty} u(t-a)\,e^{-st}\,dt$$

$$= \{\text{see P.7.2 and P.7.3}\} = \frac{1}{s} - \frac{1}{s}\;e^{-as} = \frac{1}{s}\big(1-e^{-as}\big)$$

7.13. Given time domain function sinusoidal function

$$f(t) = \sin(at)\,u(t) = \frac{e^{jat}-e^{-jat}}{2j}\,u(t)$$

and $u(t) = 1, \quad (t \geq 0)$ then, by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{+\infty} \frac{e^{jat}-e^{-jat}}{2j}\,u(t)\,e^{-st}\,dt = \frac{1}{2j}\int_0^{+\infty} e^{(ja-s)t}\,dt$$

$$-\frac{1}{2j}\int_0^{+\infty} e^{-(ja+s)t}\,dt$$

$\{$see P.7.11$\}$

$$= \frac{1}{2j}\left(\frac{1}{s-ja} - \frac{1}{s+ja}\right) = \frac{1}{2j}\left(\frac{(s+ja)-(s-ja)}{(s-ja)(s+ja)}\right) = \frac{1}{2j}\,\frac{2ja}{s^2+a^2} = \frac{a}{s^2+a^2}$$

7.14. Given time domain function

$$f(t) = \cos(at)\,u(t) = \frac{e^{jat} + e^{-jat}}{2}\,u(t)$$

and $u(t) = 1, \quad (t \geq 0)$ then, by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{+\infty} \frac{e^{jat} + e^{-jat}}{2}\,e^{-st}\,dt = \frac{1}{2}\int_0^{+\infty} e^{(ja-s)t}\,dt + \frac{1}{2}\int_0^{+\infty} e^{-(ja+s)t}\,dt$$

{see P.7.11}

$$= \frac{1}{2}\left(\frac{1}{s-ja} + \frac{1}{s+ja}\right) = \frac{1}{2}\left(\frac{(s+ja)+(s-ja)}{(s-ja)(s+ja)}\right) = \frac{1}{2}\frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2}$$

7.15. Given time–domain *periodic* function

$$\forall t \in \mathbb{R} : f(t) = f(t+T)$$

by taking advantage of the time–shifting property, it follows that

$$F(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt = \int_0^{T} e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

{change variable: $t = z, t = z+T, t = z+2T, \ldots \;\; \therefore \;\; dt = dz, \ldots$}

$$= \int_0^{T} e^{-sz} f(z)\,dz + \int_T^{2T} e^{-s(z+T)} f(z+t)\,dz + \int_{2T}^{3T} e^{-s(z+2T)} f(z+2T)\,dz + \cdots$$

$\{\mathscr{L}\{f(t-a)\} = e^{-as}F(s)\}$

$$= \int_0^{T} e^{-sz} f(z)\,dz + e^{-sT}\int_0^{T} e^{-sz} f(z)\,dz + e^{-s2T}\int_0^{T} e^{-sz} f(z)\,dz + \cdots$$

$$= \left(1 + e^{-sT} + e^{-s2T} + \cdots\right)\int_0^{T} e^{-sz} f(z)\,dz$$

{sum of infinite geometric series: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ (see Ch. 1)}

{return to the original variable}

$$= \frac{1}{1-e^{-sT}}\int_0^{T} e^{-st} f(t)\,dt$$

7.2 Basic Properties

7.16. Given first derivative $f'(x)$ as

$$g(t) = f'(t) = \frac{df(t)}{dt}$$

and by recalling that condition for existence of the transform is that $f(t)$ is finite, i.e. $\big(f(t) < \infty\big)$ for all t, then by definition, Laplace transformation $G(s) = \mathscr{L}\big\{g(t)\big\}$ is

$$G(s) = \int_0^{+\infty} g(t)\, e^{-st}\, dt = \int_0^{+\infty} f'(t)\, e^{-st}\, dt = \int_0^{+\infty} e^{-st}\, \frac{df(t)}{\cancel{dt}}\, \cancel{dt}$$

$$\Big\{\text{integration by parts: } u = e^{-st},\quad dv = df(t)\quad \therefore\quad du = -s\, e^{-st},\quad v = \int df(t) = f(t)\Big\}$$

$$= f(t)\, e^{-st}\Big|_0^{+\infty} + s \underbrace{\int_0^{+\infty} f(t)\, e^{-st}\, dt}_{F(s)=\mathscr{L}\{f(t)\}} = \Big[\cancel{f(+\infty)\, e^{-\infty}}^{\,0} - f(0)\, \cancel{e^{0}}^{\,1}\Big] + s\, F(s)$$

$$= s\, F(s) - f(0)$$

because $\big(f(t) < \infty \quad \therefore \quad f(+\infty)\cdot 0 = 0\big)$. In addition, $f(0)$ stands for the initial condition of $f(t)$ at $t = 0$ (i.e. just before the signal is injected into system).

Reminder: Laplace transformation of a function's first derivative is

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = s\, F(s) - f(0)$$

This general case is known as "first derivative property" of Laplace transformation.

7.17. Given second derivative $f''(x)$ as

$$g(t) = f''(t) = \frac{d^2 f(t)}{dt^2} = \frac{df'(t)}{dt}$$

by definition, Laplace transformation $G(s) = \mathscr{L}\big\{g(t)\big\}$ is

$$G(s) = \int_0^{+\infty} g(t)\, e^{-st}\, dt = \int_0^{+\infty} f''(t)\, e^{-st}\, dt = \int_0^{+\infty} e^{-st}\, \frac{df'(t)}{\cancel{dt}}\, \cancel{dt}$$

$$\Big\{\text{integration by parts: } u = e^{-st},\quad dv = df'(t)\quad \therefore\quad du = -s\, e^{-st},\quad v = \int df'(t) = f'(t)\Big\}$$

$$= f'(t)\, e^{-st}\Big|_0^{+\infty} + s \int_0^{+\infty} f'(t)\, e^{-st} dt = \Big[\cancel{f'(+\infty)\, e^{-\infty}}^{\,0} - f'(0)\, \cancel{e^{0}}^{\,1}\Big]$$

$$+ s \underbrace{\int_0^{+\infty} f'(t)\, e^{-st}\, dt}_{\text{see P.7.16}}$$

$$= s\big(s\, F(s) - f(0)\big) - f'(0) = s^2\, F(s) - s\, f(0) - f'(0)$$

7.18. Given third derivative $f'''(x)$ as

$$g(t) = f'''(t) = \frac{d^3 f(t)}{dt^3} = \frac{df''(t)}{dt}$$

by definition, Laplace transformation $G(s) = \mathscr{L}\{g(t)\}$ is

$$G(s) = \int_0^{+\infty} g(t)\, e^{-st}\, dt = \int_0^{+\infty} f'''(t)\, e^{-st}\, dt = \int_0^{+\infty} e^{-st}\, \frac{df''(t)}{dt}\, dt$$

$$\left\{\text{integration by parts: } u = e^{-st},\ \ dv = df''(t)\ \ \therefore\ \ du = -s\, e^{-st},\ \ v = \int df''(t) = f''(t)\right\}$$

$$= f''(t)\, e^{-st}\Big|_0^{+\infty} + s \int_0^{+\infty} f''(t)\, e^{-st} dt = \Big[\underbrace{f''(+\infty)\, e^{-\infty}}_{0} - f''(0)\, \underbrace{e^{0}}_{1}\Big]$$

$$+ s \underbrace{\int_0^{+\infty} f''(t)\, e^{-st}\, dt}_{\text{see P.7.17}}$$

$$= s\left(s^2 F(s) - s\, f(0) - f'(0)\right) - f''(0) = s^3 F(s) - s^2 f(0) - s\, f'(0) - f''(0)$$

Reminder: For the reference, this derivative property of Laplace transformation is generalized for nth derivative as

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n \mathscr{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

7.19. Given composite sinusoidal function

$$f(t) = \sin(2t) = \frac{e^{2jt} - e^{-2jt}}{2j}$$

by definition (see P.7.13), Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} \frac{e^{2jt} - e^{-2jt}}{2j}\, e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^{+\infty} e^{(2j-s)t}\, dt - \frac{1}{2j} \int_0^{+\infty} e^{-(2j+s)t}\, dt$$

$$\{\text{see P.7.11}\}$$

$$= \frac{1}{2j}\left(\frac{1}{s-2j} - \frac{1}{s+2j}\right) = \frac{1}{2j}\, \frac{(s+2j) - (s-2j)}{(s-2j)(s+2j)} = \frac{1}{2j}\, \frac{2\,2j}{s^2 + 2^2} = \frac{2}{s^2+4}$$

7.20. Given sinusoidal function

$$f(t) = \cos(2t) = \frac{e^{2jt} + e^{-2jt}}{2}$$

by definition (see P.7.14), Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} \frac{e^{2jt} + e^{-2jt}}{2}\, e^{-st}\, dt = \frac{1}{2}\int_0^{+\infty} e^{(2j-s)t}\, dt + \frac{1}{2}\int_0^{+\infty} e^{-(2j+s)t}\, dt$$

$\{$see P.7.11$\}$

$$= \frac{1}{2}\left(\frac{1}{s-2j} + \frac{1}{s+2j}\right) = \frac{1}{2}\,\frac{(s+2j)+(s-2j)}{(s-2j)(s+2j)} = \frac{1}{2}\,\frac{2s}{s^2+2^2} = \frac{s}{s^2+4}$$

7.21. Given sinusoidal function multiplied by power of t

$$f(t) = t\sin(2t)$$

Method 1: by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} t\sin(2t)\, e^{-st}\, dt$$

$\left\{\text{integration by parts: } u = te^{-st},\ \ dv = \sin(2t)\,dt\ \ \therefore\ \ du = e^{-st}(1-st)\,dt,\ \ v = -\frac{1}{2}\cos(2t)\right\}$

$$= -\frac{1}{2}\cos(2t)\, t\, e^{-st}\Big|_0^{+\infty} + \frac{1}{2}\int_0^{+\infty} \cos(2t)\, e^{-st}(1-st)\, dt$$

$$= 0 + \frac{1}{2}\underbrace{\int_0^{+\infty} \cos(2t)\, e^{-st}\, dt}_{\mathscr{L}\{\cos(2t)\},\ \text{P.7.20}} - \frac{1}{2}\, s \int_0^{+\infty} \cos(2t)\, e^{-st}\, t\, dt$$

$$= \frac{1}{2}\,\frac{s}{s^2+4} - \frac{1}{2}\, s \int_0^{+\infty} \cos(2t)\, e^{-st}\, t\, dt$$

because

$$\cos(2t)\, t\, e^{-st}\Big|_0^{+\infty} = \lim_{t\to\infty}\big(\cos(2t)\, t\, e^{-st}\big) - \cos(2(0))\,(0)\, e^{-s(0)}$$

where the above limit may be proved, for example, by the Squeeze theorem. In addition,

$$\int_0^{+\infty} \cos(2t)\, e^{-st}\, t\, dt =$$

$\left\{\text{integration by parts: } u = te^{-st},\ \ dv = \cos(2t)\,dt\ \ \therefore\ \ du = e^{-st}(1-st)dt,\ \ v = \frac{1}{2}\sin(2t)\right\}$

$$= \frac{1}{2}\, te^{-st}\, \sin(2t)\Big|_0^{+\infty} - \frac{1}{2}\int_0^{+\infty} \sin(2t) e^{-st}(1-st)\, dt$$

$$= 0 - \frac{1}{2}\underbrace{\int_0^{+\infty} \sin(2t)\, e^{-st}\, dt}_{\mathscr{L}\{\sin(2t)\},\ \text{P.7.19}} + \frac{1}{2}\, s \int_0^{+\infty} t\, \sin(2t)\, e^{-st}\, dt = -\frac{1}{2}\,\frac{2}{s^2+4}$$

$$+ \frac{1}{2}\, s \underbrace{\int_0^{+\infty} t\, \sin(2t)\, e^{-st}\, dt}_{F(s)}$$

$$= -\frac{1}{\cancel{2}}\,\frac{\cancel{2}\,1}{s^2+4} + \frac{1}{2}\, s\, F(s)$$

because

$$\sin(2t)\, t\, e^{-st}\Big|_0^{+\infty} = \lim_{t\to\infty}\big(\sin(2t)\, t\, e^{-st}\big) - \sin(2(0))\,(0)\, e^{-s(0)}$$

($\lim_{t\to\infty}(\sin(2t)\, t\, e^{-st}) \to 0$; $\sin(2(0))\,(0)\, e^{-s(0)} \to 0$)

Altogether,

$$F(s) = \frac{1}{2}\,\frac{s}{s^2+4} - \frac{1}{2}\, s \int_0^{+\infty} \cos(2t)\, e^{-st}\, t\, dt = \frac{1}{2}\,\frac{s}{s^2+4} - \frac{1}{2}\, s\left[-\frac{1}{s^2+4} + \frac{1}{2}\, s\, F(s)\right]$$

$$= \frac{1}{2}\,\frac{s}{s^2+4} + \frac{1}{2}\,\frac{s}{s^2+4} - \frac{1}{4}\, s^2\, F(s)$$

$$F(s)\,\frac{s^2+4}{4} = \frac{s}{s^2+4} \quad \therefore \quad F(s) = \frac{4s}{(s^2+4)^2}$$

Method 2: After some practice, the property of multiplication of a function by powers of t (see P.7.6) may be adapted directly, as

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

In this example, power of t^1 is $n = 1$, so multiplying with power of t property is applied directly as

$$f(t) = \sin(2t) \quad \therefore \quad F(s) = \frac{2}{s^2+4} \quad \therefore \quad F^{(1)}(s) = \frac{dF(s)}{ds} = -\frac{4s}{(s^2+4)^2} \quad \text{and} \quad (-1)^1 = -1$$

$$\therefore$$

$$\mathscr{L}\{t\, \sin(2t)\} = (-1)^1\, \frac{dF(s)}{ds} = \frac{4s}{(s^2+4)^2}$$

7.22. Given sinusoidal function multiplied by power of t as

$$f(t) = t^2 \cos t$$

derivation by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is a bit longer than in P.7.21. Taking advantage of the property of multiplication of a function by powers of t,

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

Here, power of t^2 is $n = 2$ thus $(-1)^2 = 1$, so "multiplying with power of t" property leads to

$$f(t) = \cos t \quad \therefore \quad F(s) = \frac{s}{s^2+1}$$

$$\therefore \quad F^{(2)}(s) = \frac{d^2F(s)}{ds^2} = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right) = \frac{d}{ds}\left(\frac{-s^2+1}{(s^2+1)^2}\right) = \frac{2s(s^2-3)}{(s^2+1)^3}$$

$$\therefore$$

$$\mathscr{L}\left\{t^2 \cos t\right\} = (-1)^2 \frac{d^2F(s)}{ds^2} = \frac{2s(s^2-3)}{(s^2+1)^3}$$

which is, evidently, much faster than derivation by definition.

7.23. Given function multiplied by exponent

$$g(t) = e^{at} f(t)$$

by definition, Laplace transformation is

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$\therefore$$

$$F(s-a) = \int_0^{+\infty} f(t)\, e^{-(s-a)t}\, dt = \int_0^{+\infty} f(t)\, e^{at}\, e^{-st} dt = e^{at} \int_0^{+\infty} f(t)\, e^{-st} dt$$
$$= e^{at}\, \mathscr{L}\{f(t)\}$$

Reminder: This general relation is known as "frequency shifting" property

$$F(s-a) = e^{at}\, \mathscr{L}\{f(t)\}$$

Note that the shifting of s argument by a generates exponential term e^{at} that multiples the non–shifted function's transformation.

7.24. Given function $f(t)$ whose argument is time–scaled by a as

$$g(t) = f(at)$$

by definition, Laplace transformation $F(s) = \mathscr{L}\left\{f(t)\right\}$ is

$$\mathscr{L}\{f(t)\} = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$\therefore$$

$$\mathscr{L}\{f(at)\} = \int_0^{+\infty} f(at)\, e^{-st}\, dt = \int_0^{+\infty} f(at)\, e^{-st}\, \frac{a}{a}\, dt = \int_0^{+\infty} f(at)\, e^{-st}\, \frac{1}{a}\, d(at)$$

$$= \frac{1}{a} \int_0^{+\infty} f(at)\, e^{-st}\, d(at) = \left\{ \text{change variable: } z = \frac{s}{a} \quad \therefore \quad s = a\, z \right\}$$

$$= \frac{1}{a} \int_0^{+\infty} f(at)\, e^{-z(at)}\, d(at) = \frac{1}{a} \mathscr{L}\{f(z)\}$$

$$= \frac{1}{a} \mathscr{L}\left\{ f\left(\frac{s}{a}\right) \right\} = \frac{1}{a}\, F\left(\frac{s}{a}\right)$$

Note that basic idea in this proof is to replace all arguments t with arguments at. By doing so, form of Laplace transformation relative to variable z is achieved.

Reminder: This general relation is known as "time scaling" property of Laplace transformation

$$\mathscr{L}\{f(at)\} = \frac{1}{a}\, F\left(\frac{s}{a}\right)$$

7.25. Given function $f(t)$ that is scaled by t as

$$g(t) = \frac{f(t)}{t}$$

assuming that $\lim_{t \to 0} g(t)$ exists. Note that $1/t = t^{-1}$, then by definition and the multiplying with power of t property, Laplace transformation is derived as

$$f(t) = t\, g(t) \quad \therefore \quad \mathscr{L}\{t\, g(t)\} = \mathscr{L}\{f(t)\}$$

$$\therefore$$

$$(-1)^1 \frac{d}{ds}\Big[\mathscr{L}\{g(t)\}\Big] = F(s)$$

$$\int d\Big[\mathscr{L}\{g(t)\}\Big] = -\int F(s)\, ds$$

$$\therefore \quad \{\text{change to temporary variable } s = z\}$$

$$\mathscr{L}\{g(t)\} = -\int_{\infty}^{s} F(z)\, dz = \int_s^{\infty} F(z)\, dz$$

$$\therefore$$

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(z)\, dz$$

Reminder: This general relation of division by t is also known as "frequency–domain integration" property of Laplace transformation

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(z)\,dz$$

where z is temporary variable that helps to preserve s as the final result variable.

7.26. Given time–domain function $g(t)$ as

$$g(t) = \int_0^t f(z)\,dz$$

Method 1: by definition, Laplace transformation $F(s) = \mathscr{L}\{f(t)\}$ is

$$G(s) = \int_0^{+\infty} g(t)\,e^{-st}\,dt = \mathscr{L}\left\{\int_0^t f(z)\,dz\right\} = \int_0^{+\infty}\left[\int_0^t f(z)\,dz\right] e^{-st}\,dt$$

$$\left\{\text{integration by parts: } u = \int_0^t f(z)\,dz,\quad v' = e^{-st}\quad \therefore\quad u' = f(t),\quad v = -\frac{1}{s}\,e^{-st}\right\}$$

$$= \left[-\frac{1}{s}\,e^{-st}\int_0^t f(z)\,dz\right]_0^{\infty} - \int_0^{\infty} -\frac{1}{s}\,e^{-st} f(t)\,dt$$

$$= \left[-\frac{1}{s}\,\cancel{e^{-\infty}}^{\,0}\int_0^{+\infty} f(z)\,dz\right]_0^{\infty} + \frac{1}{s}\underbrace{\int_0^{\infty} f(t)e^{-st}\,dt}_{F(s)=\mathscr{L}\{f(t)\}}$$

$\therefore$

$$\mathscr{L}\left\{\int_0^t f(z)\,dz\right\} = \frac{F(s)}{s}$$

Method 2: taking advantage of derivative property of Laplace transformation,

$$g(t) = \int_0^t f(z)\,dz\quad \therefore\quad g'(t) = f(t),\quad \text{and,}\quad g(0) = 0$$

$$\mathscr{L}\{g'(t)\} = \mathscr{L}\{f(t)\} = F(s)$$

where,

$$\mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0) = s\mathscr{L}\{g(t)\}$$

$\therefore$

$$F(s) = s\mathscr{L}\left\{\int_0^t f(z)\,dz\right\}$$

$\therefore$

$$\mathscr{L}\left\{\int_0^t f(z)\,dz\right\} = \frac{F(s)}{s}$$

Reminder: This general relation also known as "time-domain integration" property of Laplace transformation

$$\mathscr{L}\left\{\int_0^t f(z)\,dz\right\} = \frac{F(s)}{s}$$

where z is temporary variable that helps to preserve t as the final result variable.

7.27. Given time domain function $f(t)$ as

$$f(t) = \int_0^t (z^2 + z - e^z)\,dz$$

by taking advantage of the transformation properties as

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\int_0^t (z^2 + z - e^z)\,dz\right\} = \mathscr{L}\left\{\int_0^t z^2\,dz\right\} + \mathscr{L}\left\{\int_0^t z\,dz\right\} - \mathscr{L}\left\{\int_0^t e^z\,dz\right\}$$

it follows that,

$$\mathscr{L}\{p^2\} = \frac{2}{s^3} \ \{\text{see P.7.7}\} \quad \therefore \quad \mathscr{L}\left\{\int_0^t z^2\,dz\right\} = \frac{2/s^3}{s} = \frac{2}{s^4}$$

$$\mathscr{L}\{p\} = \frac{1}{s^2} \ \{\text{see P.7.6}\} \quad \therefore \quad \mathscr{L}\left\{\int_0^t z\,dz\right\} = \frac{1/s^2}{s} = \frac{1}{s^3}$$

$$\mathscr{L}\{e^p\} = \frac{1}{s-1} \ \{\text{see P.7.9}\} \quad \therefore \quad \mathscr{L}\left\{\int_0^t e^z\,dz\right\} = \frac{1/(s-1)}{s} = \frac{1}{s(s-1)}$$

$$\therefore$$

$$\mathscr{L}\{f(t)\} = \frac{2}{s^4} + \frac{1}{s^3} - \frac{1}{s(s-1)} = \frac{-s^3 + s^2 + s - 2}{s^4(s-1)}$$

7.28. Given time–domain function

$$f(t) = \frac{e^t - e^{-3t}}{t}$$

by taking advantage of the transformation properties as

$$\mathscr{L}\left\{\frac{e^t - e^{-3t}}{t}\right\} = \mathscr{L}\left\{\frac{e^t}{t}\right\} - \mathscr{L}\left\{\frac{e^{-3t}}{t}\right\}$$

it follows that

$$\mathscr{L}\left\{e^{t}\right\} = \frac{1}{s-1} \quad \{\text{see P.7.9}\}$$

$$\mathscr{L}\left\{e^{-3t}\right\} = \frac{1}{s+3} \quad \{\text{see P.7.9}\}$$

$$\therefore$$

$$\mathscr{L}\left\{\frac{e^{t}-e^{-3t}}{t}\right\} = \int_{s}^{\infty} \frac{1}{z-1}\,dz - \int_{s}^{\infty} \frac{1}{z+3}\,dz \quad \{\text{see P.7.25}\}$$

$$= \ln(z-1)\Big|_{s}^{\infty} - \ln(z+3)\Big|_{s}^{\infty} = \ln\frac{z-1}{z+3}\Big|_{s}^{\infty} = \ln\frac{z-1}{z+3}\Big|^{\infty} - \ln\frac{z-1}{z+3}\Big|_{s}$$

$$= \left\{\frac{\infty}{\infty}\right\} - \ln\frac{s-1}{s+3} \overset{\text{l'H}}{=} \lim_{s\to\infty} \ln\frac{1}{1}^{\,0} - \ln\frac{s-1}{s+3} = -\ln\frac{s-1}{s+3} = \ln\left(\frac{s-1}{s+3}\right)^{-1}$$

$$= \ln\frac{s+3}{s-1}$$

7.29. Given time–domain function

$$f(t) = (t-2)^3\,u(t-2)$$

Reminder:

$$\mathscr{L}\left\{t^{n}u(t)\right\} = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}\left\{f(t-a)\right\} = e^{-as}F(s)$$

Method 1: by taking advantage of transformation properties, first, transformation of the the non–delayed version of $f(t)$ simply $f(t) = t^3 \;\;\therefore\;\; (n=3)$, thus

$$f(t) = t^3 \;\;\therefore\;\; F(s) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

then, delaying $f(t)$ by $a = 2$ results in

$$f(t) = (t-1)^3 \;\;\therefore\;\; F(s) = e^{-2s}\frac{6}{s^4}$$

Method 2: by definition,

$$
\begin{aligned}
F(s) &= \int_0^{+\infty} f(t)\, e^{-st}\, dt \\
&= \int_0^2 (t-2)^3\, e^{-st}\, dt + \int_2^{+\infty} (t-2)^3\, e^{-st}\, dt = 0 + \int_2^{+\infty} (t-2)^3\, e^{-st}\, dt \\
&\left\{\text{change variable: } z = t-2,\ dz = dt,\ t = z+2\right\} \\
&\left\{\text{adjust the limits: } t \to 2 \Rightarrow z \to 0,\ t \to \infty \Rightarrow z \to \infty\right\} \\
&= \int_0^{+\infty} z^3\, e^{-s(z+2)}\, dz = \int_0^{+\infty} z^3\, e^{-2s}\, e^{-sz}\, dz = e^{-2s} \int_0^{+\infty} z^3\, e^{-sz}\, dz \\
&= e^{-2s}\, \mathscr{L}\left\{z^3\right\} \quad \text{(see P.7.8)} \\
&= e^{-2s}\, \frac{6}{s^4}
\end{aligned}
$$

7.30. Given time–domain function $\left(\text{a.k.a sinc}\,(t)\right)$

$$\frac{\sin t}{t}$$

Reminder:

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(z)\, dz$$

$$\mathscr{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

by taking advantage of transformation's frequency–domain integration properties, first, transformation of $f(t)$ version that is not divided by t,

$$
\begin{aligned}
F(s) = \mathscr{L}\{\sin(t)\} &= \frac{1}{s^2+1} \\
&\therefore \\
\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^{\infty} F(z)\, dz = \int_s^{\infty} \frac{1}{z^2+1}\, dz = \arctan(z)\Big|_s^{\infty} = \arctan(\infty) - \arctan(s) \\
&= \frac{\pi}{2} - \arctan(s) \\
&\left\{\text{trig identity: } \arctan(s) + \arctan\left(\frac{1}{s}\right) = \frac{\pi}{2},\quad (s > 0)\right\} \\
&= \arctan\left(\frac{1}{s}\right)
\end{aligned}
$$

7.3 Inverse Laplace Transformation

7.31. For example,
Method 1: Inverse of $1/s$ may be derived, for example, by using result in P.7.8 where it is found that

$$\mathscr{L}\left\{t^a u(t)\right\} = \frac{a!}{s^{a+1}}$$

which after setting $a = 0$ reduces to $\mathscr{L}\left\{t^0\, u(t)\right\} = \mathscr{L}\{u(t)\}$ (because $t^0 = 1$), so that

$$\mathscr{L}\{u(t)\} = \frac{0!}{s^{0+1}} = \frac{1}{s} \quad \Rightarrow \quad f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = u(t)$$

Method 2: See P.7.1 and P.7.2, as $t \geq 0$, then $f(t) = \mathscr{L}^{-1}\{1/s\} = u(t)$.

7.32. See P.7.3, as $a = 2$, then $f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\, e^{-2s}\right\} = u(t-2)$

7.33. See P.7.4, $f(t) = \mathscr{L}^{-1}\{1\} = \delta(t)$

7.34. See P.7.5, as $a = 1$, then $f(t) = \mathscr{L}^{-1}\left\{e^{-s}\right\} = \delta(t-1)$

7.35. See P.7.6, $f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\, u(t)$

7.36. See P.7.7, $f(t) = \mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2\, u(t)$.

7.37. See P.7.14, as $a = 3$ then $f(t) = \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos(3\,t)\, u(t)$

7.38. See P.7.12, as $a = 2$ then $f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\left(1 - e^{-2s}\right)\right\} = u(t) - u(t-2)$

7.39. See P.7.21, $f(t) = \mathscr{L}^{-1}\left\{\frac{4s}{(s^2+4)^2}\right\} = t\,\sin(2t)$

7.40. See P.7.30, $\mathscr{L}^{-1}\left\{\arctan\left(\frac{1}{s}\right)\right\} = \frac{\sin t}{t}$

7.41. See P.7.11, $\mathscr{L}^{-1}\left\{\frac{2s}{s^2-a^2}\right\} = \left(e^{at} + e^{-at}\right) u(t)$

7.42. Inverse LT of rational function are found by partial fractions and decomposition of rational fractions into already known rational terms. For example,

$$f(t) = \frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} + \frac{2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$
$$= \frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}$$

Therefore,

$$\mathscr{L}^{-1}\{f(t)\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$$
$$- 2\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$
$$= \{\text{see P.7.13, P.7.14 and P.7.23}\}$$
$$= e^{-2t}\cos(t) - 2\,e^{-2t}\sin(t) = e^{-2t}(\cos(t) - 2\sin(t))$$

7.4 Differential Equations

7.43. Given first order differential equation with constant coefficients

$$y' + 1 = 0 \quad \therefore \quad \frac{dy(t)}{dt} + 1 = 0$$

Method 1: postulate that $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT derivative properties as

$$y' + 1 = 0 \quad \therefore \quad \mathscr{L}\{y' + 1\} = \mathscr{L}\{0\} \quad \therefore \quad \mathscr{L}\{y'\} + \mathscr{L}\{1\} = \mathscr{L}\{0\}$$
$$\big(sY(s) - y(0)\big) + \frac{1}{s} = 0 \quad \therefore \quad sY(s) = -\frac{1}{s} \quad \therefore \quad Y(s) = -\frac{1}{s^2}$$

Reminder:

$$\mathscr{L}\{t\} = \frac{1}{s^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

Inverse Laplace, see P.7.6, is then

$$y(t) = -t$$

Method 2: direct integration results in

$$\frac{dy(t)}{dt} + 1 = 0 \quad \therefore \quad \frac{dy(t)}{dt} = -1 \quad \therefore \quad \int dy(t) = -\int dt \quad \therefore \quad y(t) = -t + C \quad \big(y(0) = 0\big)$$
$$\therefore$$

$$y(0) = -(0) + C = 0 \quad \therefore \quad C = 0$$

$$\therefore$$

$$y(t) = -t$$

Verification:

$$y(t) = -t$$

$$y'(t) = -1$$

$$\therefore$$

$$-1 + 1 = 0 \quad \checkmark$$

7.44. Given differential equation with constant coefficients

$$-2y' + y = 0$$

Method 1: assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$-2y' + y = 0 \quad \therefore \quad \mathscr{L}\left\{-2y' + y\right\} = \mathscr{L}\{0\} \quad \therefore \quad -2\mathscr{L}\left\{y'\right\} + \mathscr{L}\{y\} = \mathscr{L}\{0\}$$

$$-2\big(sY(s) - y(0)\big) + Y(s) = 0 \quad \therefore \quad -2sY(s) + Y(s) + 2 = 0 \quad \therefore \quad Y(s) = \frac{-2}{-2s+1} = \frac{\cancel{-2}}{\cancel{-2}}\,\frac{1}{s-1/2}$$

Reminder:

$$\mathscr{L}\left\{e^{t/2}\right\} = \frac{1}{s-1/2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s-1/2}\right\} = e^{t/2}$$

Inverse transformation results in (see P.7.9)

$$y(t) = e^{t/2}$$

Method 2: direct integration of differential equation with separable variables

$$-2\frac{dy}{dt} + y = 0 \quad \therefore \quad \frac{dy}{dt} = \frac{1}{2}\,y \quad \therefore \quad \int \frac{dy}{y} = \frac{1}{2}\int dt$$

$$\therefore$$

$$\ln y(t) + \ln C = \frac{t}{2} \quad \therefore \quad \ln\big(C\,y(t)\big) = \frac{t}{2} \quad \therefore \quad y(t) = C\,e^{t/2} \quad \therefore \quad y(0) = C\,\cancel{e^{0/2}}^{\,1} = C = 1$$

$$\therefore$$

$$y(t) = e^{t/2}$$

Verification:

$$y(t) = e^{t/2}$$

$$y'(t) = \frac{1}{2}\, e^{t/2}$$

$$\therefore$$

$$-\cancel{2}\,\frac{1}{\cancel{2}}\, e^{t/2} + e^{t/2} = 0 \quad \checkmark$$

7.45. **Method 1:** Given differential equation with constant coefficients

$$y'' - 1 = 0$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$y'' - 1 = 0 \;\; \therefore \;\; \mathscr{L}\left\{y'' - 1\right\} = \mathscr{L}\{0\} \;\; \therefore \;\; \mathscr{L}\left\{y''\right\} - \mathscr{L}\{1\} = \mathscr{L}\{0\}$$

$$\left(s^2\, Y(s) - s\, y(0) - y'(0)\right) - \frac{1}{s} = 0 \;\; \therefore \;\; s^2\, Y(s) - \frac{1}{s} = 0 \;\; \therefore \;\; Y(s) = \frac{1}{s^3}$$

Reminder:

$$\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3} \;\; \Rightarrow \;\; \mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

Inverse transformation (see P.7.7) results in

$$Y(s) = \frac{1}{s^3}\,\frac{2}{2} = \frac{1}{2}\,\frac{2}{s^3} \;\; \therefore \;\; y(t) = \frac{1}{2}\, t^2$$

Method 2: direct integration results in

$$y'' - 1 = 0$$

$$\frac{dy'}{dt} - 1 = 0 \;\; \therefore \;\; \int dy' = \int dt \;\; \therefore \;\; y'(t) = t + C_1 \;\; \therefore \;\; y'(0) = 0 + C_1 = 0 \;\; \therefore \;\; C_1 = 0$$

$$\therefore \;\; y'(t) = t$$

$$\therefore$$

$$y' = \frac{dy}{dt} = t \;\; \therefore \;\; \int dy = \int t\, dt \;\; \therefore \;\; y(t) = \frac{t^2}{2} + C_2 \;\; \therefore \;\; y(0) = \frac{0^2}{2} + C_2 = 0 \;\; \therefore \;\; C_2 = 0$$

$$y(t) = \frac{1}{2}\, t^2$$

Verification:

$$y(t) = \frac{t^2}{2}$$
$$y'(t) = t$$
$$y''(t) = 1$$
$$\therefore$$
$$y'' - 1 = 1 - 1 = 0 \ \checkmark$$

7.46. Given differential equation with constant coefficients

$$y'' - y' + 1 = 0$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$y'' - y' + 1 = 0 \ \ \therefore \ \ \mathscr{L}\left\{y'' - y' + 1\right\} = \mathscr{L}\{0\}$$
$$\mathscr{L}\left\{y''\right\} - \mathscr{L}\left\{y'\right\} + \mathscr{L}\{1\} = \mathscr{L}\{0\}$$
$$\left(s^2\,Y(s) - s\,y(0) - y'(0)\right) - \left(sY(s) - y(0)\right) + \frac{1}{s} = 0$$
$$\left(s^2\,Y(s) - s - 1)\right) - \left(sY(s) - 1)\right) + \frac{1}{s} = 0$$
$$s^2\,Y(s) - sY(s) = s - \frac{1}{s}$$
$$Y(s)\left(s(s-1)\right) = \frac{s^2 - 1}{s}$$
$$Y(s) = \frac{\cancel{(s-1)}(s+1)}{s^2\cancel{(s-1)}} = \frac{s+1}{s^2}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation.

$$Y(s) = \frac{s+1}{s^2} = \frac{A}{s} + \frac{B}{s^2} = \frac{As + B}{s^2}$$

which is to say (by equality of the numerators)

$$A = 1 \quad \text{and} \quad B = 1$$
$$\therefore$$
$$Y(s) = \frac{1}{s} + \frac{1}{s^2}$$

Reminder:

$$\mathscr{L}\{1\} = \frac{1}{s} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathscr{L}\{t\} = \frac{1}{s^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

Inverse Laplace transformation is

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} \quad \therefore \quad y(t) = 1 + t$$

Verification:

$$\begin{aligned} y(t) &= 1 + t \\ y'(t) &= 1 \\ y''(t) &= 0 \\ &\therefore \\ y'' - y' + 1 &= 0 - 1 + 1 = 0 \quad \checkmark \end{aligned}$$

7.47. Given differential equation with constant coefficients

$$y'' + 2y' + 2y = 0$$

assume $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$\begin{aligned} y'' + 2y' + 2y = 0 \quad \therefore \quad \mathscr{L}\left\{y'' + 2y' + 2y\right\} &= \mathscr{L}\{0\} \\ \mathscr{L}\left\{y''\right\} + 2\mathscr{L}\left\{y'\right\} + 2\mathscr{L}\{y\} &= \mathscr{L}\{0\} \\ \left(s^2 Y(s) - s\,y(0) - y'(0)\right) + 2\big(sY(s) - y(0)\big) + 2Y(s) &= 0 \\ \left(s^2 Y(s) + s - 1)\right) + 2\big(sY(s) + 1)\big) + 2Y(s) &= 0 \\ s^2 Y(s) + 2sY(s) + 2Y(s) &= -(s+1) \\ Y(s)\big(s^2 + 2s + 2\big) &= -(s+1) \\ Y(s) &= \frac{-s + -1}{s^2 + 2s + 2} \end{aligned}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation. Denominator of $Y(s)$ has complex roots, $s_1 = -1+i$ and $s_2 = -1-i$. It follows that

$$Y(s) = \frac{-s-1}{s^2+2s+2} = \frac{-s-1}{(s-s_1)(s-s_2)} = \frac{A}{s-s_1} + \frac{B}{s-s_2} = \frac{A(s-s_2)+B(s-s_1)}{(s-s_1)(s-s_2)}$$
$$= \frac{s(A+B) - As_2 - Bs_1}{(s-s_1)(s-s_2)}$$

which is to say (by equality of the left and right side numerators)

$$A + B = -1 \quad \therefore \quad \underline{B = -1 - A}$$
$$-As_2 - Bs_1 = -1 \quad \therefore \quad As_2 + Bs_1 = 1$$
$$\therefore$$
$$As_2 + (-1-A)s_1 = 1 \quad \therefore \quad A = \frac{1+s_1}{s_2-s_1} = \frac{1+(-1+i)}{-1-i-(-1+i)} = \frac{i}{-2i} = -\frac{1}{2}$$
$$\therefore \quad B = -1 + \frac{1}{2} = -\frac{1}{2}$$

Reminder:

$$\mathscr{L}\left\{e^{-st}\right\} = \frac{1}{s+1} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-st}$$

Inverse Laplace transformation is

$$Y(s) = -\frac{1}{2}\frac{1}{s-s_1} - \frac{1}{2}\frac{1}{s-s_2}$$
$$y(t) = -\frac{1}{2}e^{s_1 t} - \frac{1}{2}e^{s_2 t} = -\frac{1}{2}e^{(-1+i)t} - \frac{1}{2}e^{(-1-i)t} = -\frac{1}{2e^t}e^{it} - \frac{1}{2e^t}e^{-it}$$
$$= -\frac{1}{2e^t}\left(\cos t + i\sin t\right) - \frac{1}{2e^t}\left(\cos t - i\sin t\right) = -e^{-t}\cos t$$

Verification:

$$y(t) = -e^{-t}$$
$$y'(t) = e^{-t}$$
$$y''(t) = -e^{-t}$$
$$\therefore$$
$$-e^{-t} + 2e^{-t} - 2e^{-t} = 2e^{-t} - 2e^{-t} = 0 \quad \checkmark$$

7.48. Given non–homogenous differential equation with constant coefficients

$$y'' + y = \sin(2t)$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$\begin{aligned}
y'' + y = \sin(2t) \quad \therefore \quad \mathscr{L}\left\{y'' + y\right\} &= \mathscr{L}\{\sin(2t)\} \\
\mathscr{L}\left\{y''\right\} + \mathscr{L}\{y\} &= \mathscr{L}\{\sin(2t)\} \\
\left(s^2\,Y(s) - s\,y(0) - y'(0)\right) + Y(s) &= \frac{2}{s^2+4} \\
s^2\,Y(s) + Y(s) &= \frac{2}{s^2+4} \\
Y(s)(s^2+1) &= \frac{2}{s^2+4} \\
Y(s) &= \frac{2}{(s^2+1)(s^2+4)}
\end{aligned}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation. Denominator of $Y(s)$ has complex roots, $s_1 = +i$, $s_2 = -i$, $s_3 = +2i$, and $s_4 = -2i$. It follows that

$$\begin{aligned}
Y(s) &= \frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} = \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+4)} \\
&= \frac{s^3(A+C) + s^2(B+D) + s(4A+C) + 4B + D}{(s^2+1)(s^2+4)}
\end{aligned}$$

which is to say (by equality of the numerators)

$$\left.\begin{aligned}
A + C &= 0 \\
B + D &= 0 \\
4A + C &= 0 \\
4B + D &= 2
\end{aligned}\right\} \quad \therefore \quad A = 0,\, B = \frac{2}{3},\, C = 0,\, D = -\frac{2}{3}$$

Reminder:

$$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

Inverse Laplace transformation is (see P.7.13)

$$\begin{aligned}
Y(s) &= \frac{2}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4} \\
y(t) &= \frac{2}{3}\sin t - \frac{1}{3}\sin(2t)
\end{aligned}$$

Verification:

$$\left.\begin{aligned} y(t) &= \tfrac{2}{3}\sin t - \tfrac{1}{3}\sin(2t) \\ y'(t) &= \tfrac{2}{3}\cos t - \tfrac{2}{3}\cos(2t) \\ y''(t) &= -\tfrac{2}{3}\sin t + \tfrac{4}{3}\sin(2t) \end{aligned}\right\} \quad \therefore \quad -\frac{2}{3}\sin t + \frac{4}{3}\sin(2t) + \frac{2}{3}\sin t - \frac{1}{3}\sin(2t) = \sin(2t) \quad \checkmark$$

7.49. Given non–homogenous differential equation with constant coefficients

$$y'' + y' = 2e^{-t}$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$\begin{aligned} y'' + y' = 2e^{-t} \quad \therefore \quad \mathscr{L}\left\{y'' + y'\right\} &= \mathscr{L}\left\{2e^{-t}\right\} \\ \mathscr{L}\left\{y''\right\} + \mathscr{L}\left\{y'\right\} &= \mathscr{L}\left\{2e^{-t}\right\} \\ \left(s^2\,Y(s) - s\,y(0) - y'(0)\right) + \left(sY(s) - y(0)\right) &= \frac{2}{s+1} \\ s^2\,Y(s) - s - 1 + sY(s) - 1 &= \frac{2}{s+1} \\ Y(s)(s^2 + s) - s - 2 &= \frac{2}{s+1} \\ Y(s) &= \frac{(s+2)(s+1)+2}{(s^2+s)(s+1)} = \frac{s^2+3s+4}{s(s+1)(s+1)} \end{aligned}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation. It follows that

$$\begin{aligned} Y(s) &= \frac{s^2+3s+4}{s(s+1)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{A(s^2+2s+1) + Bs(s+1) + Cs}{s(s+1)^2} \\ &= \frac{s^2(A+B) + s(2A+B+C) + A}{s(s+1)(s+1)} \end{aligned}$$

which is to say (by equality of the numerators)

$$\left.\begin{aligned} A + B &= 1 \\ 2A + B + C &= 3 \\ A &= 4 \end{aligned}\right\} \quad \therefore \quad A = 4,\, B = -3,\, C = -2$$

so that inverse Laplace transformation is derived as

$$\begin{aligned} Y(s) &= \frac{4}{s} - \frac{3}{s+1} - \frac{2}{(s+1)^2} \\ y(t) &= 4 - 3e^{-t} - 2te^{-t} \end{aligned}$$

Verification:

$$\left.\begin{aligned} y(t) &= 4 - 3e^{-t} - 2te^{-t} \\ y'(t) &= 2te^{-t} + e^{-t} \\ y''(t) &= -2te^{-t} + e^{-t} \end{aligned}\right\} \quad \therefore \quad -\cancel{2te^{-t}} + e^{-t} + \cancel{2te^{-t}} + e^{-t} = 2e^{-t} \quad \checkmark$$

7.50. Given non–homogenous differential equation with constant coefficients

$$y'' - y' = \cos(t)$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$\begin{aligned} y'' - y' = \sin(t) \quad \therefore \quad \mathscr{L}\left\{y'' - y'\right\} &= \mathscr{L}\{\cos(t)\} \\ \mathscr{L}\left\{y''\right\} - \mathscr{L}\left\{y'\right\} &= \mathscr{L}\{\cos(t)\} \\ \left(s^2\,Y(s) - s\,y(0) - y'(0)\right) - \left(sY(s) - y(0)\right) &= \frac{s}{s^2+1} \\ s^2\,Y(s) - sY(s) &= \frac{s}{s^2+1} \\ Y(s) &= \frac{\cancel{s}\,1}{\cancel{s}(s-1)(s^2+1)} = \frac{1}{(s-1)(s^2+1)} \end{aligned}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation. Denominator of $Y(s)$ has complex roots, $s_1 = +i$ and $s_2 = -i$. It follows that

$$\begin{aligned} Y(s) = \frac{1}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1) + (Bs+C)(s-1)}{(s-1)(s^2+1)} \\ &= \frac{s^2(A+B) + s(C-B) + A - C}{(s-1)(s^2+1)} \end{aligned}$$

which is to say (by equality of the numerators)

$$\left.\begin{aligned} A + B &= 0 \\ C - B &= 0 \\ A - C &= 1 \end{aligned}\right\} \quad \therefore \quad A = \frac{1}{2},\ B = -\frac{1}{2},\ C = -\frac{1}{2}$$

Reminder:

$$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

Inverse Laplace transformation is (see P.7.14)

$$Y(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}$$
$$y(t) = \frac{1}{2}e^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)$$

Verification:

$$\left.\begin{aligned} y(t) &= \tfrac{1}{2}e^t - \tfrac{1}{2}\cos(t) - \tfrac{1}{2}\sin(t) \\ y'(t) &= \tfrac{1}{2}e^x + \tfrac{1}{2}\sin(x) - \tfrac{1}{2}\cos(x) \\ y''(t) &= \tfrac{1}{2}e^x + \tfrac{1}{2}\cos(x) + \tfrac{1}{2}\sin(x) \end{aligned}\right\} \quad \therefore$$

$$\frac{1}{2}e^x + \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) - \frac{1}{2}e^x - \frac{1}{2}\sin(x) + \frac{1}{2}\cos(x) = \cos(t) \quad \checkmark$$

7.51. Given non–homogenous differential equation with constant coefficients

$$y''' - y'' = 0$$

assuming $Y(s) = \mathscr{L}\{y(t)\}$, then by taking advantage of LT properties as

$$y''' - y'' = 0 \quad \therefore \quad \mathscr{L}\left\{y''' - y''\right\} = \mathscr{L}\{0\}$$
$$\mathscr{L}\left\{y'''\right\} - \mathscr{L}\left\{y''\right\} = \mathscr{L}\{0\}$$
$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - \left(s^2\,Y(s) - s\,y(0) - y'(0)\right) = 0$$
$$s^3Y(s) + 3s^2 - s - 4 - s^2\,Y(s) - 3s + 1 = 0$$
$$Y(s) = \frac{-3s^2 + 4s + 3}{s^2(s-1)}$$

Rational function $Y(s)$ must be decomposed by partial fraction method, so that it is possible to find its inverse Laplace transformation. It follows that

$$Y(s) = \frac{-3s^2 + 4s + 3}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} = \frac{As(s-1) + B(s-1) + Cs^2}{s^2(s-1)}$$
$$= \frac{s^2(A+C) + s(B-A) - B}{s^2(s-1)}$$

which is to say (by equality of the numerators)

$$\left.\begin{aligned} A + C &= -3 \\ B - A &= 4 \\ -B &= 3 \end{aligned}\right\} \quad \therefore \quad A = -7, B = -3, C = 4$$

so that inverse Laplace transformation is derived as

$$Y(s) = -\frac{7}{s} - \frac{3}{s^2} + \frac{4}{s-1}$$
$$y(t) = -7 - 3t + 4e^t$$

Verification:

$$\left.\begin{aligned} y(t) &= -7 - 3t + 4e^t \\ y'(t) &= 4e^t - 3 \\ y''(t) &= 4e^t \\ y'''(t) &= 4e^t \end{aligned}\right\} \quad \therefore \quad 4e^t - 4e^t = 0 \quad \checkmark$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41203-5